Online Reviews and Collaborative Service Provision: A Signal-Jamming Model

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Online Appendices

S1 Proofs

We first prove a lemma that is useful in proving the results throughout the paper.

Lemma A.1. *When either V or R is observed as a signal S, given any effort level x and y, a positive signal always leads to a higher posterior belief of the service provider being a high type than a negative signal, i.e.,* $\alpha_1(x,y) > \alpha_0(x,y)$, $\forall x,y \in [0,1]$.

Proof. Recall that the posterior belief can be derived according to Bayes' rule as

$$\begin{aligned}\alpha_1(x,y) &= \frac{\Pr(S=1|H,x,y)}{\Pr(S=1|H,x,y)+\Pr(S=1|L,x,y)} = \frac{1}{1+\frac{\Pr(S=1|L,x,y)}{\Pr(S=1|H,x,y)}} \\ \alpha_0(x,y) &= \frac{\Pr(S=0|H,x,y)}{\Pr(S=0|H,x,y)+\Pr(S=0|L,x,y)} = \frac{1}{1+\frac{\Pr(S=0|L,x,y)}{\Pr(S=0|H,x,y)}}\end{aligned}$$

Note that as long as $\Pr(S=1|L,x,y) < \Pr(S=1|H,x,y)$, we have $\frac{\Pr(S=1|L,x,y)}{\Pr(S=1|H,x,y)} < \frac{1-\Pr(S=1|L,x,y)}{1-\Pr(S=1|H,x,y)} = \frac{\Pr(S=0|L,x,y)}{\Pr(S=0|H,x,y)}$, which is equivalent to $\alpha_1(x,y) > \alpha_0(x,y)$. Clearly, $\Pr(S=1|L,x,y) < \Pr(S=1|H,x,y)$ holds whether signal S is V or R for $\forall x,y \in [0,1]$ throughout the paper (in both the baseline model and the extended model). □

Proof of Proposition 1:

Proof. According to Equation (9), the first order derivative with respect to y is a constant. Because $\triangle\alpha^V\left(x^V,y^V\right) > 0$ (by Lemma A.1), the service provider sets the optimal effort level of the client as $y^V = 1$. Consequently, the optimal effort level of the provider x^V is the solution to the first order condition (by Equation (8)) $\frac{1+\mu}{4}\triangle\alpha^V\left(x^V,1\right) = 2cx^V$. Denote

$$K(x) = \frac{1+\mu}{4}\triangle\alpha^V(x,1) = \frac{1-\mu}{2\left[3-\mu-(1+\mu)x\right]} \tag{A.1}$$

after substituting α_1^V and α_0^V into the expression of $\triangle\alpha^V(x,1)$, and denote $C(x) = 2cx$. Hence, an interior solution x^V arises when $K(x)$ and $C(x)$ intersect within $(0,1)$. Notice that $K(x)$ is continuous and increasing in x for $x \in [0,1]$, and $K(0) = \frac{1-\mu}{2(3-\mu)} > 0 = C(0)$. Therefore, when $K(1) < C(1)$, that is, $c > \frac{1}{8}$, $K(x)$ intersects $C(x)$ within $(0,1)$. Furthermore, such an intersection is unique because when $c > \frac{1}{8}$, $K(x) = C(x)$ yields a unique solution within $(0,1)$, that is, $x^V = \frac{c(3-\mu)-\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)}$. It can be easily verified that the expression within the square root is positive, because $c^2(3-\mu)^2 - c\left(1-\mu^2\right) > 0$ if and only if $c > \frac{1-\mu^2}{(3-\mu)^2}$ and $\frac{1-\mu^2}{(3-\mu)^2} \leq \frac{1}{8}$. □

Proof of Corollary 1:

Proof. Rewrite $x^V = \frac{(3-\mu)-\sqrt{(3-\mu)^2-(1-\mu^2)/c}}{2(1+\mu)}$. It is thus easy to see that x^V decreases in c. To show x^V decreases in μ, take the first order derivative of x^V with respect to μ such that

$$\frac{\partial x^V}{\partial \mu} = \frac{12c - 1 - \mu(1+4c) - 4\sqrt{c^2(3-\mu)^2 - c(1-\mu^2)}}{2(1+\mu)^2\sqrt{c^2(3-\mu)^2 - c(1-\mu^2)}}.$$

If the first part of the numerator is negative (i.e., $12c - 1 - \mu(1+4c) < 0$), then $\frac{\partial x^V}{\partial \mu} < 0$ holds immediately. If $12c - 1 - \mu(1+4c) > 0$, then $12c - 1 - \mu(1+4c) - 4\sqrt{c^2(3-\mu)^2 - c(1-\mu^2)} < 0$ if and only if $[12c - 1 - \mu(1+4c)]^2 - 16\left[c^2(3-\mu)^2 - c\left(1-\mu^2\right)\right] < 0$, which holds because $[12c - 1 - \mu(1+4c)]^2 - 16\left[c^2(3-\mu)^2 - c\left(1-\mu^2\right)\right] = -(8c-1)(1+\mu)^2 < 0$ when $c > \frac{1}{8}$. □

In order to prove Proposition 2, we first prove a useful lemma. Define

$$M(x;y) \equiv \frac{1+\mu}{4(1+w)}\left[\alpha_1^R(x,y) - \alpha_0^R(x,y)\right],$$

where

$$\begin{cases} \alpha_1^R(x,y) = \frac{(x+y)+2w\left(1-y^2\right)}{(1+\mu)(x+y)+4w(1-y^2)} \\ \alpha_0^R(x,y) = \frac{2(1+w)-(x+y)-2w\left(1-y^2\right)}{4(1+w)-(1+\mu)(x+y)-4w(1-y^2)} \end{cases}.$$

Lemma A.2. *(1) $M(x;y)$ is increasing in x for $\forall x,y \in [0,1]$; and (2) $\frac{\partial^2}{\partial x^2}M(x;y)$ is increasing in x for $\forall x,y \in [0,1]$.*

Proof. (1) To show $\frac{\partial}{\partial x}M(x;y)=\frac{(1+\mu)}{4(1+w)}\left[\frac{\partial}{\partial x}\alpha_1^R(x,y)-\frac{\partial}{\partial x}\alpha_0^R(x,y)\right]>0$, note that:

$$\frac{\partial}{\partial x}\alpha_1^R(x,y)=\frac{2w(1-\mu)\left(1-y^2\right)}{\left[4w\left(1-y^2\right)+(1+\mu)(x+y)\right]^2}\geq 0$$

$$\frac{\partial}{\partial x}\alpha_0^R(x,y)=\frac{-2(1-\mu)\left(1+wy^2\right)}{\left[4-(1+\mu)(x+y)+4wy^2\right]^2}<0$$

for $\forall x,y\in[0,1]$ and $\mu\in(0,1)$. Therefore, $M(x;y)$ is increasing in x.

(2) Note that $\frac{\partial^2}{\partial x^2}M(x;y)=\frac{(1+\mu)}{4(1+w)}\left[\frac{\partial^2}{\partial x^2}\alpha_1^R(x,y)-\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)\right]$, and

$$\frac{\partial^2}{\partial x^2}\alpha_1^R(x,y)=\frac{-4w\left(1-\mu^2\right)\left(1-y^2\right)}{\left[4w\left(1-y^2\right)+(1+\mu)(x+y)\right]^3}$$

$$\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)=\frac{-4\left(1-\mu^2\right)\left(1+wy^2\right)}{\left[4-(1+\mu)(x+y)+4wy^2\right]^3}.$$

It is easy to see that the denominator of the above $\frac{\partial^2}{\partial x^2}\alpha_1^R(x,y)$ is positive and increasing in x. In addition, its numerator is negative and independent of x. It follows that $\frac{\partial^2}{\partial x^2}\alpha_1^R(x,y)$ is increasing in x. Similarly, the numerator of the above $\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)$ is negative and independent of x. To check the sign of the denominator of $\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)$, note that $4-(1+\mu)(x+y)+4wy^2>4-2(1+\mu)>0$, because $x+y\leq 2$ and $\mu<1$. Therefore, the denominator of $\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)$ is positive and decreasing in x. Consequently, $\frac{\partial^2}{\partial x^2}\alpha_0^R(x,y)$ is decreasing in x. It follows that $\frac{\partial^2}{\partial x^2}M(x)$, being proportional to $\frac{\partial^2}{\partial x^2}\alpha_1^R-\frac{\partial^2}{\partial x^2}\alpha_0^R$, is increasing in x. □

Proof of Proposition 2:

Proof. First, note that (10) simplifies to $\Pr\left\{R=1|\theta,x^R,y^R\right\}=\frac{\mu_\theta\left(x^R+y^R\right)+2w\left(1-\left(y^R\right)^2\right)}{2(1+w)}$, where $\theta\in\{H,L\}$, $\mu_H=1$ and $\mu_L=\mu$. Then, we have :

$$\begin{cases}\alpha_1^R=\Pr\left\{\theta=H|R=1,x^R,y^R\right\}=\frac{\left(x^R+y^R\right)+2w\left[1-\left(y^R\right)^2\right]}{(1+\mu)\left(x^R+y^R\right)+4w\left[1-\left(y^R\right)^2\right]}\\ \alpha_0^R=\Pr\left\{\theta=H|R=0,x^R,y^R\right\}=\frac{2(1+w)-\left(x^R+y^R\right)-2w\left[1-\left(y^R\right)^2\right]}{4(1+w)-(1+\mu)\left(x^R+y^R\right)-4w\left[1-\left(y^R\right)^2\right]}\end{cases}. \tag{A.2}$$

Similar to section 4.1, (5) can be simplified to the following:

$$\left(x^R,y^R\right)=\arg\max_{0\leq x,y\leq 1}\ \alpha_0^R+\frac{1}{2}\left(\alpha_1^R-\alpha_0^R\right)\frac{(1+\mu)(x+y)+4w\left(1-y^2\right)}{2(1+w)}-cx^2,$$

where α_1^R and α_0^R are given in (A.2). Take the first derivative of the above objective function (which we denote as $E\pi$) with respect to x and y, respectively, we get:

$$\begin{array}{rcl} \frac{\partial E\pi}{\partial x} & = & \frac{1+\mu}{4(1+w)}\triangle\alpha^R\left(x^R, y^R\right) - 2cx \\ \frac{\partial E\pi}{\partial y} & = & \frac{\triangle\alpha^R\left(x^R, y^R\right)}{4(1+w)}\left(1+\mu-8wy\right) \end{array}.$$

From Lemma A.1, we know $\triangle\alpha^R\left(x^R, y^R\right) > 0$ for any equilibrium effort levels. Hence, $\frac{\partial E\pi}{\partial y} = 0$ implies the equilibrium effort level for the client is:

$$y^R = \begin{cases} \frac{1+\mu}{8w} & \text{if } w \geq \frac{1+\mu}{8} \\ 1 & \text{otherwise} \end{cases} \tag{A.3}$$

Denote $M(x) \equiv M\left(x; y^R\right) = \frac{1+\mu}{4(1+w)}\triangle\alpha^R\left(x, y^R\right)$. Consequently, the optimal effort level for the provider, x^R, is the fixed-point solution to the first order condition $M(x) - 2cx = 0$, whenever such an interior solution exists within $[0,1]$. Next, we will show that there exists a unique solution to the above first order condition when $c > \frac{1}{2}M(1)$.

Denote $C(x) = 2cx$. Then $C(0) = 0$ and $C(x)$ is increasing in x. It is easy to verify that $M(0) > 0 = C(0)$. From part (1) of Lemma (A.2), we know $M(x)$ is also increasing in x. Then, whether $M(x)$ intersects $C(x)$ and how many times they intersect depend on the concavity (and convexity) of $M(x)$ and the value of $M(1)$. The concavity of $M(x)$ depends on the sign of second order derivative of $M(x)$ with respect to x. Note that there are two mutually exhaustive and exclusive cases for the value of $\frac{\partial^2}{\partial x^2}M(x)|_{x=0}$: (1) $\frac{\partial^2}{\partial x^2}M(x)|_{x=0} < 0$ and (2) $\frac{\partial^2}{\partial x^2}M(x)|_{x=0} \geq 0$.

In case (1), because part (2) of Lemma (A.2) states that $\frac{\partial^2}{\partial x^2}M(x)$ is increasing in x, then if $\frac{\partial^2}{\partial x^2}M(x)|_{x=0} < 0$, we know that $\frac{\partial^2}{\partial x^2}M(x)$ either changes sign once or never for $x \in [0,1]$. In the case when it changes sign at $\hat{x} \in (0,1)$, we must have $M(x)$ is concave in x for $x \leq \hat{x}$ (due to negative second order derivative in this range) and convex in x for $x \geq \hat{x}$ (due to positive second order derivative). In the case when it never changes sign, we know $M(x)$ is concave in x for $x \in [0,1]$. In case (2), because part (2) of Lemma (A.2) states that $\frac{\partial^2}{\partial x^2}M(x)$ is increasing in x, then if $\frac{\partial^2}{\partial x^2}M(x)|_{x=0} > 0$, we know that $\frac{\partial^2}{\partial x^2}M(x)$ can never change sign. Thus, $M(x)$ is convex in x due to the positive second order derivative. To summarize, $M(x)$ is either concave or convex for all $x \in [0,1]$ or it is concave for $x < \hat{x}$ and convex for $x > \hat{x}$.

Knowing the concavity (and convexity) of $M(x)$ and $M(0) > 0$, it follows that if $C(1) = 2c > M(1)$, then $M(x)$ intersects $C(x)$ *exactly* once in the range of $x \in [0, 1]$. In addition, the intersection point is the fixed point solution to $M(x) = 2cx$, i.e., x^R. □

Proof of Corollary 2:

Proof. From (A.3), it is straightforward to see that for $w > \frac{1+\mu}{8}$, y^R is decreasing in w and independent of c. For $c > \frac{1}{2}M(1)$, since x^R is the unique solution to $M(x) = 2cx$ (Proposition 2), we know x^R is decreasing in c because $M(x)$ is independent of c and $2cx$ is decreasing in c. Similarly, if we can show $M(x)$ is decreasing in w, then x^R must be decreasing in w. Next, we will prove that $M(x)$ is decreasing in w in the following two cases: (1) when $w > \frac{1+\mu}{8}$ and (2) when $w \leq \frac{1+\mu}{8}$.

Case (1): When $w > \frac{1+\mu}{8}$, we must have $y^R = \frac{1+\mu}{8w}$. Substitute this y^R into (A.2), we get:

$$\begin{cases} \alpha_1^R\left(x, y^R = \frac{1+\mu}{8w}\right) = \frac{64w^2+3+\mu(2-\mu)+32wx}{2[64w^2+(1+\mu)^2+(1+\mu)16wx]} \\ \alpha_0^R\left(x, y^R = \frac{1+\mu}{8w}\right) = \frac{64w-(3-\mu)(1+\mu)-32wx}{2[64w-(1+\mu)^2-16w(1+\mu)x]} \end{cases}$$

It is straightforward to verify that α_1^R decreases in w and α_0^R increases in w by taking the first derivative of them with respect to w:

$$\frac{\partial \alpha_1^R}{\partial w} = \frac{-8(1-\mu)\left[16w(1+\mu) + (1+\mu)^2 x + 64w^2 x\right]}{\left[(1+\mu)^2 + 16wx^R(1+\mu) + 64w^2\right]^2} < 0$$

$$\frac{\partial \alpha_0^R}{\partial w} = \frac{8(1-\mu)\left[(1+\mu)\left(8-(1+\mu)x\right)\right]}{\left[(1+\mu)^2 + 16w(1+\mu)x^R - 64w\right]^2} > 0.$$

Therefore, $M\left(x; y^R = \frac{1+\mu}{8w}\right) = \left[\alpha_1^R - \alpha_0^R\right]\frac{1+\mu}{4(1+w)}$ is decreasing in w.

Case (2): When $w \leq \frac{1+\mu}{8}$, we must have $y^R = 1$. Substitute this y^R into (A.2), we get:

$$M\left(x; y^R = 1\right) = \left[\alpha_1^R - \alpha_0^R\right]\frac{1+\mu}{4(1+w)} = \frac{1-\mu}{8(1+w) - 2(1+\mu)(1+x)}, \tag{A.4}$$

which is decreasing in w. To summarize, $M\left(x; y^R\right)$ is decreasing in w for any $w \geq 0$.

To see how x^R and y^R change in μ, first, it is obvious that $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$ is weakly increasing in μ. When $w \leq \frac{1+\mu}{8}$ so that $y^R = 1$, take the first derivative of (A.4) with respect to x,

$$\frac{\partial}{\partial \mu} M\left(x; y^R = 1\right) = \frac{-(1-x) - 2w}{\left(3 + 4w - x - \mu - \mu x\right)^2} < 0.$$

Therefore, $M\left(x; y^R = 1\right)$ is decreasing in μ, so x^R is decreasing in μ. When $w > \frac{1+\mu}{8}$ so that $y^R = \frac{1+\mu}{8w}$, x^R may change in μ non-monotonically, as Figure 5 illustrates. $\square$

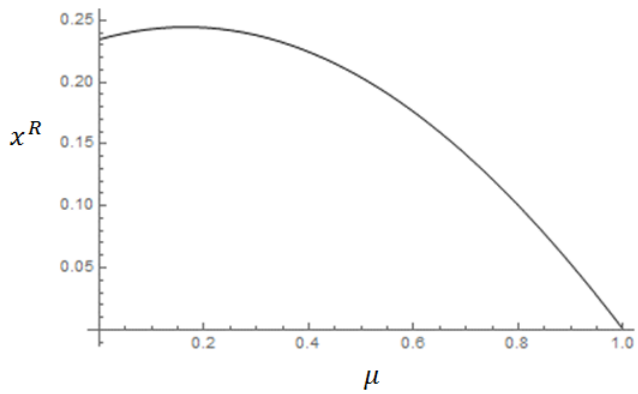


Figure 5: x^R Changes in μ ($w = 0.25$, $c = 0.15$)

Proof of Proposition 3:

Proof. $y^R \le 1 = y^V$ comes directly from comparing $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$ to $y^V = 1$. Clearly, the inequality is strict for any $w > \frac{1+\mu}{8}$.

In what follows, we first prove that $\frac{1}{2}M(1) < \frac{1}{8}$, so as long as $c > \frac{1}{8}$, x^R is the unique solution to $M(x) = 2cx$. From the proof of Corollary 2, we know that $M(x)$ is decreasing in w, i.e., $M\left(1|_{w>0}\right) < M\left(1|_{w=0}\right)$. When $w = 0$, we have $y^R|_{w=0} = 1$. Recall, when $y^R = 1$, $M(x)$ simplifies to (A.4). It follows that

$$M\left(x|_{w=0}\right) = \frac{1-\mu}{2\left[3-\mu-(1+\mu)x\right]},$$

from which we get $M\left(1|_{w=0}\right) = \frac{1}{4}$. Therefore, we have $\frac{1}{2}M(1) \le \frac{1}{8}$ and the inequality is strict for any $w > 0$.

Next, we will show that when $w = 0$, $x^R = x^V$. To see this, note that the above $M\left(x|_{w=0}\right)$ is the same as $K(x)$ defined in (A.1). From the proof of Proposition 1, we know that x^V is the unique solution to $K(x) = 2cx$ when $c > \frac{1}{8}$. Therefore, x^V is also the unique solution to $M\left(x|_{w=0}\right) = 2cx$, i.e., when $w = 0$, $x^R = x^V$. Finally, combing this $x^R|_{w=0} = x^V$ with the fact that x^R is decreasing in w (Corollary 2), we know that $x^R \le x^V$ and the inequality is strict for any $w > 0$. $\square$

Proof of Proposition 4:

Proof. Note that $EV = \frac{(1+\mu)\left(x^S+y^S\right)}{4}$, where $S = V$ or R. The rest follows directly from applying the results in Proposition 3 (i.e., $x^R \leq x^V$ and $y^R \leq y^V$). $\square$

Proof of Proposition 5:

Proof. Because $y^e = \min\left\{\frac{1+\mu}{8w}, 1\right\}$, $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$ and $y^V = 1$, we have $y^R = y^e \leq y^V$ and the inequality is strict for $w > \frac{1+\mu}{8}$. $\square$

In order to prove Proposition 6, we first prove a useful lemma. Define

$$\tilde{M}\left(x;y\right) \equiv \frac{\sigma_H + \sigma_L\mu}{4\left(1+w\right)}\left[\tilde{\alpha}_1^R\left(x,y\right) - \tilde{\alpha}_0^R\left(x,y\right)\right],$$

where

$$\begin{cases} \tilde{\alpha}_1^R\left(x,y\right) = \frac{\sigma_H(x+y)+2\sigma_H w\left(1-y^2\right)}{(\sigma_H+\sigma_L\mu)(x+y)+2(\sigma_H+\sigma_L)w(1-y^2)} \\ \tilde{\alpha}_0^R\left(x,y\right) = \frac{2(1+w)-\sigma_H(x+y)-2\sigma_H w\left(1-y^2\right)}{4(1+w)-(\sigma_H+\sigma_L\mu)(x+y)-2w(\sigma_H+\sigma_L)(1-y^2)} \end{cases}.$$

Lemma A.3. *(1) $\tilde{M}\left(x;y\right)$ is increasing in x for $\forall x, y \in [0,1]$; and (2) $\frac{\partial^2}{\partial x^2}\tilde{M}\left(x;y\right)$ is increasing in x for $\forall x, y \in [0,1]$.*

Proof. (1) To show $\frac{\partial}{\partial x}\tilde{M}\left(x;y\right) = \frac{\sigma_H+\sigma_L\mu}{4(1+w)}\left[\frac{\partial}{\partial x}\tilde{\alpha}_1^R\left(x,y\right) - \frac{\partial}{\partial x}\tilde{\alpha}_0^R\left(x,y\right)\right] > 0$, note that

$$\frac{\partial}{\partial x}\tilde{\alpha}_1^R\left(x,y\right) = \frac{2w\sigma_H\sigma_L\left(1-\mu\right)\left(1-y^2\right)}{\left[2w\left(1-y^2\right)\left(\sigma_H+\sigma_L\right) + \left(\sigma_H+\mu\sigma_L\right)\left(x+y\right)\right]^2}$$

$$\frac{\partial}{\partial x}\tilde{\alpha}_0^R\left(x,y\right) = \frac{-2\left[\left(1+w\right)\left(\sigma_H-\mu\sigma_L\right) - 2w\left(1-y^2\right)\left(1-\mu\right)\sigma_H\sigma_L\right]}{\left[4\left(1+w\right) - 2w\left(\sigma_H+\sigma_L\right)\left(1-y^2\right) - \left(\sigma_H+\mu\sigma_L\right)\left(x+y\right)\right]^2}$$

Clearly, $\frac{\partial}{\partial x}\tilde{\alpha}_1^R\left(x|y\right) \geq 0$ because $\sigma_H, \sigma_L \in (0,1]$ and $\mu \in (0,1)$. Denote the numerator of the above $\frac{\partial}{\partial x}\tilde{\alpha}_0^R\left(x,y\right)$ as $-2 \cdot N_0$. $\frac{\partial}{\partial x}\tilde{\alpha}_0^R\left(x|y\right) < 0$ because N_0 is positive. To show $N_0 > 0$, rewrite N_0 as the following:

$$\begin{aligned}
N_0 &= (1+w)\,\sigma_H\left(1-\frac{\sigma_L}{\sigma_H}\mu\right) - w\left(1-y^2\right)(1-\mu)\,\sigma_H\sigma_L \\
&\geq (1+w)\,\sigma_H(1-\mu) - w\left(1-y^2\right)(1-\mu)\,\sigma_H\sigma_L \\
&= \sigma_H(1-\mu)\left[1+(1-\sigma_L)\,w+\sigma_L w y^2\right] \\
&> 0,
\end{aligned}$$

where the first inequality is from substituting $\frac{\sigma_L}{\sigma_H} \leq 1$ and the last inequality is from $\sigma_H, \sigma_L \in (0,1]$ and $\mu \in (0,1)$.

(2) Note that $\frac{\partial^2}{\partial x^2}\tilde{M}(x;y) = \frac{\sigma_H+\sigma_L\mu}{4(1+w)}\left[\frac{\partial^2}{\partial x^2}\tilde{\alpha}_1^R(x,y) - \frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)\right]$, and

$$\frac{\partial^2}{\partial x^2}\tilde{\alpha}_1^R(x,y) = \frac{-4w\left(1-y^2\right)(1-\mu)\,\sigma_H\sigma_L(\sigma_H+\sigma_L\mu)}{\left[2w(\sigma_H+\sigma_L)(1-y^2)+(\sigma_H+\sigma_L\mu)(x+y)\right]^3}$$

$$\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y) = \frac{-4(\sigma_H+\sigma_L\mu)\left[(1+w)(\sigma_H-\sigma_L\mu) - w\left(1-y^2\right)(1-\mu)\,\sigma_H\sigma_L\right]}{\left[4(1+w)-2w(\sigma_H+\sigma_L)(1-y^2)-(\sigma_H+\sigma_L\mu)(x+y)\right]^3}$$

It is easy to see that the denominator of the above $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_1^R(x,y)$ is positive and increasing in x. In addition, its numerator is negative and independent of x. It follows that $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_1^R(x,y)$ is increasing in x. Notice that the numerator of the above $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$ can be written as $-4(\sigma_H+\sigma_L\mu)\,N0$, where $N0$ is the same as that defined in part(1) of the proof. Because $N0$ is shown to be positive, the numerator of $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$ is negative and independent of x. To check the sign of its denominator, we re-write the terms inside the bracket of the denominator of $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$ as:

$$\begin{aligned}
& 4-(\sigma_H+\sigma_L\mu)(x+y)+4w-2w(\sigma_H+\sigma_L)+2w(\sigma_H+\sigma_L)\,y^2 \\
\geq\; & 4-(\sigma_H+\sigma_L\mu)(x+y) \\
>\; & 0,
\end{aligned}$$

where the first inequality is from $\sigma_H+\sigma_L \leq 2$ and the last inequality is from $\sigma_H+\sigma_L\mu < 2$ and $x+y \leq 2$. Consequently, the denominator of $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$ is positive and decreasing in x. It follows that $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$ is decreasing in x. Thus, $\frac{\partial^2}{\partial x^2}M(x)$, being proportional to $\frac{\partial^2}{\partial x^2}\tilde{\alpha}_1^R(x,y) - \frac{\partial^2}{\partial x^2}\tilde{\alpha}_0^R(x,y)$, is increasing in x. □

Proof of Proposition 6:

Proof. Substituting (15) into (10), we get:

$$\Pr(R=1|\theta,\tilde{x}^R,\tilde{y}^R) = \sigma_\theta \frac{\mu_\theta\left(\tilde{x}^R+\tilde{y}^R\right)+2w\left(1-\left(\tilde{y}^R\right)^2\right)}{2(1+w)},$$

where $\theta \in \{H, L\}$, $\mu_H = 1$ and $\mu_L = \mu$. Clearly, $\Pr(R=0|\theta,\tilde{x}^R,\tilde{y}^R) = 1-\Pr(R=0|\theta,\tilde{x}^R,\tilde{y}^R)$. Thus, we get the following $\tilde{\alpha}_1^R$ and $\tilde{\alpha}_0^R$:

$$\begin{cases} \tilde{\alpha}_1^R = \frac{\sigma_H\left(\tilde{x}^R+\tilde{y}^R\right)+2\sigma_H w\left(1-\left(\tilde{y}^R\right)^2\right)}{(\sigma_H+\sigma_L\mu)(\tilde{x}^R+\tilde{y}^R)+2(\sigma_H+\sigma_L)w\left(1-(\tilde{y}^R)^2\right)} \\ \tilde{\alpha}_0^R = \frac{2(1+w)-\sigma_H\left(\tilde{x}^R+\tilde{y}^R\right)-2\sigma_H w\left(1-\left(\tilde{y}^R\right)^2\right)}{4(1+w)-(\sigma_H+\sigma_L\mu)(\tilde{x}^R+\tilde{y}^R)-2w(\sigma_H+\sigma_L)\left(1-(\tilde{y}^R)^2\right)} \end{cases} \tag{A.5}$$

Then, the equilibrium effort levels are the solutions to the following problem:

$$\begin{aligned}\left(\tilde{x}^R,\tilde{y}^R\right) = \arg\max_{0\le x,\, y\le 1} \; & \tfrac{1}{2}\tilde{\alpha}_1^R\left\{\tfrac{\sigma_H}{1+w}\left[\tfrac{1}{2}(x+y)+w(1-y^2)\right]+\tfrac{\sigma_L}{1+w}\left[\tfrac{\mu}{2}(x+y)+w(1-y^2)\right]\right\} \\ & +\tfrac{1}{2}\tilde{\alpha}_0^R\left\{2-\tfrac{\sigma_H}{1+w}\left[\tfrac{1}{2}(x+y)+w(1-y^2)\right]-\tfrac{\sigma_L}{1+w}\left[\tfrac{\mu}{2}(x+y)+w(1-y^2)\right]\right\}-cx^2.\end{aligned}$$

Take the first derivative of the objective function (which we denote as $E\tilde{\pi}$) with respect to x and y, we have:

$$\begin{aligned}\frac{\partial E\tilde{\pi}}{\partial x} &= \frac{(\sigma_H+\sigma_L\mu)}{4(1+w)}\triangle\tilde{\alpha}^R\left(\tilde{x}^R,\tilde{y}^R\right)-2cx \\ \frac{\partial E\tilde{\pi}}{\partial y} &= \frac{\triangle\tilde{\alpha}^R\left(\tilde{x}^R,\tilde{y}^R\right)}{4(1+w)}\left[(\sigma_H+\sigma_L\mu)-4wy\left(\sigma_H+\sigma_L\right)\right]\end{aligned},$$

where $\triangle\tilde{\alpha}^R\left(\tilde{x}^R,\tilde{y}^R\right)=\tilde{\alpha}_1^R-\tilde{\alpha}_0^R$. Since $\triangle\tilde{\alpha}^R>0$ (because of Lemma A.1), then $\frac{\partial E\tilde{\pi}}{\partial y}=0$ implies:

$$\tilde{y}^R = \begin{cases} \frac{\sigma_H+\sigma_L\mu}{4w(\sigma_H+\sigma_L)} & \text{if } w > \frac{\sigma_H+\sigma_L\mu}{4(\sigma_H+\sigma_L)} \\ 1 & \text{otherwise} \end{cases}. \tag{A.6}$$

Denote $\tilde{M}(x) \equiv \tilde{M}\left(x;y^R\right) = \frac{(\sigma_H+\sigma_L\mu)}{4(1+w)}\triangle\tilde{\alpha}^R\left(x,\tilde{y}^R\right)$. Consequently, the optimal effort level for the provider, $\tilde{x}^R$, is the fixed-point solution to the first order condition $\tilde{M}(x)-2cx=0$, whenever such an interior solution exists within $[0,1]$.

The rest of the proof is similar to the proof for Proposition 2. Based on Lemma A.3, we can verify that $\tilde{M}(x)$ is either concave or convex for all $x\in[0,1]$ or there exists $\hat{x}\in(0,1)$ such that it is concave for $x < \hat{x}$ and convex for $x > \hat{x}$. In addition, we know

$$\tilde{M}(0) = \frac{(\sigma_H+\sigma_L\mu)\left[2w\left(1-y^2\right)(\sigma_H-\sigma_L)+(\sigma_H-\sigma_L\mu)y\right]}{2[2w(\sigma_H+\sigma_L)(1-y^2)+(\sigma_H+\sigma_L\mu)y][4-(\sigma_H+\sigma_L\mu)y+4w-2w(\sigma_H+\sigma_L)+2w(\sigma_H+\sigma_L)y^2]} > 0$$

It follows that if $C(1) = 2c > \tilde{M}(1)$, then $\tilde{M}(x)$ intersects $C(x)$ *exactly* once in the range of $x \in [0,1]$. In addition, the intersection point is the fixed point solution to $\tilde{M}(x) = 2cx$, i.e., $\tilde{x}^R$. Also note that if $C(1) = 2c < \tilde{M}(1)$, then $\tilde{M}(x)$ either intersects $C(x)$ twice in the range of $x \in [0,1]$ or is greater than $C(x)$ for all $x \in [0,1]$. However, in either case, $\tilde{x}^R = 1$ is one of the equilibria because $\frac{\partial E\tilde{\pi}}{\partial x}$ is positive at the corner solution $\tilde{x}^R = 1$. □

In order to prove Proposition 7, we first prove the following two useful lemmas. They are about the monotonicity of $\tilde{M}(x)$ under the two special cases examined in Proposition 7: (i) uninformative private signal such that $\sigma_H = \sigma_L = \sigma$; (ii) informative symmetric private signal such that $\sigma_H = \gamma$ and $\sigma_L = 1-\gamma$ ($\gamma \in \left[\frac{1}{2},1\right]$).

Lemma A.4. *When $\sigma_H = \sigma_L = \sigma \in (0,1]$, $\tilde{M}(x)$ is decreasing in w and increasing in σ.*

Proof. With $\sigma_H = \sigma_L = \sigma$, $\tilde{y}^R$ in Proposition 6 simplifies to:

$$\tilde{y}^R = \begin{cases} \frac{1+\mu}{8w} & \text{if } w > \frac{1+\mu}{8} \\ 1 & \text{otherwise} \end{cases}.$$

Substituting in $\sigma_H = \sigma_L = \sigma$ and $\tilde{y}^R$, we want to show $\tilde{M}\left(x;\tilde{y}^R\right)$ is decreasing in w and increasing in σ. Because $\tilde{y}^R$ also depends on w, we examine $\tilde{M}\left(x;\tilde{y}^R\right)$ in two cases: (1) when $w > \frac{1+\mu}{8}$, and (2) when $w \le \frac{1+\mu}{8}$.

Case (1),when $w > \frac{1+\mu}{8}$, we have $\tilde{y}^R = \frac{1+\mu}{8w}$. Substitute it into (A.5), we have:

$$\tilde{\alpha}_1^R\left(x,\tilde{y}^R = \frac{1+\mu}{8w}\right) = \frac{64w^2+3+\mu(2-\mu)+32wx}{2\left[64w^2+(1+\mu)^2+(1+\mu)16wx\right]}$$

$$\tilde{\alpha}_0^R\left(x,\tilde{y}^R = \frac{1+\mu}{8w}\right) = \frac{64w(1+w)-\sigma\left[(3-\mu)(1+\mu)+64w^2+32wx\right]}{2\left\{64w(1+w)-\sigma\left[(1+\mu)^2+64w^2+16w(1+\mu)x\right]\right\}}.$$

It is easy to verify that the above $\tilde{\alpha}_1^R$ decreases in w and the above $\tilde{\alpha}_0^R$ increases in w by checking their first derivative with respect to w, respectively:

$$\frac{\partial \tilde{\alpha}_1^R}{\partial w} = \frac{-8(1-\mu)\left[16w(1+\mu)+(1+\mu)^2 x+64w^2 x\right]}{[(1+\mu)^2+16wx(1+\mu)+64w^2]^2} < 0$$

$$\frac{\partial \tilde{\alpha}_0^R}{\partial w} = \frac{8\sigma(1-\mu)\left[(1+\mu)\left(8-\sigma(1+\mu)x\right)+16w\left(1-\sigma\right)\left(1+\mu\right)+64w^2\left(1-\sigma\right)x\right]}{[(1+\mu)^2\sigma+16w\sigma(1+\mu)x-64w-64w^2\left(1-\sigma\right)]^2} > 0$$

Therefore, $\tilde{M}\left(x;\tilde{y}^R=\frac{1+\mu}{8w}\right)=\left[\tilde{\alpha}_1^R-\tilde{\alpha}_0^R\right]\frac{\sigma(1+\mu)}{4(1+w)}$ decreases in w for $w>\frac{1+\mu}{8}$.

Case (2), when $w \leq \frac{1+\mu}{8}$, we have $\tilde{y}^R=1$. Substitute it into (A.5), we get

$$\tilde{M}\left(x;\tilde{y}^R=1\right)=\frac{\sigma(1-\mu)}{8(1+w)-2\sigma(1+\mu)(1+x)}, \tag{A.7}$$

which is decreasing in w for $0 \leq w \leq \frac{1+\mu}{8}$. To summarize both cases, $\tilde{M}(x)$ is decreasing in w for all $w \geq 0$.

Next, we will prove $\tilde{M}(x;\tilde{y}^R)$ is increasing in σ for all $\sigma \in (0,1]$. Note that $\tilde{y}^R$ is independent of σ and always positive. Therefore, in this proof, we can treat $\tilde{y}^R$ as a constant. With $\sigma_H=\sigma_L=\sigma$, (A.5) simplifies to

$$\tilde{\alpha}_1^R\left(x,y\right)=\frac{(x+y)+2w(1-y^2)}{(1+\mu)(x+y)+4w(1-y^2)}$$

$$\tilde{\alpha}_0^R\left(x,y\right)=\frac{2\left(1+w\right)-\sigma(x+y)-2\sigma w(1-y^2)}{4(1+w)-\sigma(1+\mu)(x+y)-4\sigma w(1-y^2)}.$$

It is easy to see that $\tilde{\alpha}_1^R$ is independent of σ. We can verify that $\tilde{\alpha}_0^R$ is decreasing in σ by checking its first derivative with respect to σ:

$$\frac{\partial}{\partial \sigma}\tilde{\alpha}_0^R(x,y)=\frac{-2\left(1+w\right)\left(1-\mu\right)\left(x+y\right)}{\left[4-\sigma\left(1+\mu\right)\left(x+y\right)+4w\left(1-\sigma\right)+4w\sigma y^2\right]^2} < 0$$

for $\mu \in (0,1)$ and $x+y>0$. It follows that $\tilde{M}\left(x;y\right)=\left[\tilde{\alpha}_1^R(x,y)-\tilde{\alpha}_0^R(x,y)\right]\frac{\sigma(1+\mu)}{4(1+w)}$ is increasing in σ for $\forall x,y \in [0,1]$. $\square$

Lemma A.5. *When $\sigma_H=\gamma$ and $\sigma_L=1-\gamma$, where $\gamma \in \left[\frac{1}{2},1\right]$, $\tilde{M}(x)$ is increasing in γ.*

Proof. Substituting $\sigma_H = \gamma$ and $\sigma_L = 1-\gamma$ into (A.6), we get

$$\tilde{y}^R = \begin{cases} \frac{\mu+(1-\mu)\gamma}{4w} & \text{if } w \geq \frac{\mu+\gamma-\mu\gamma}{4} \\ 1 & \text{if } w < \frac{\mu+\gamma-\mu\gamma}{4} \end{cases}. \tag{A.8}$$

Because $\tilde{y}^R$ depends on γ, we need to show that $\tilde{M}(x; \tilde{y}^R(\gamma), \gamma)$ is increasing in γ. By the chain rule,

$$\frac{d}{d\gamma}\tilde{M}\left(x; \tilde{y}^R(\gamma), \gamma\right) = \frac{\partial}{\partial\gamma}\tilde{M}(x; y, \gamma)\,|_{y=\tilde{y}^R} + \frac{\partial}{\partial y}\tilde{M}(x; y, \gamma)\,|_{y=\tilde{y}^R} \cdot \frac{d}{d\gamma}\tilde{y}^R(\gamma). \tag{A.9}$$

It is easy to show that $\frac{d}{d\gamma}\tilde{y}^R(\gamma) \geq 0$ according to (A.8). Next, we will show that $\frac{\partial}{\partial y}\tilde{M}(x; y, \gamma)\,|_{y=\tilde{y}^R} > 0$. Because $\tilde{M}(x; y, \gamma) = \left[\tilde{\alpha}_1^R(x, y) - \tilde{\alpha}_0^R(x, y)\right]\frac{(\mu+\gamma-\mu\gamma)}{4(1+w)}$, we start by examining the partial derivative of $\tilde{\alpha}_1^R$ and $\tilde{\alpha}_0^R$ with respect to y evaluated at $y = \tilde{y}^R$. It is easy to confirm that when $\sigma_H = \gamma$ and $\sigma_L = 1-\gamma$, (A.5) simplify to:

$$\tilde{\alpha}_1^R(x, y) = \frac{\gamma(x+y) + 2\gamma w(1-y^2)}{(\gamma+\mu-\gamma\mu)(x+y) + 2w(1-y^2)}$$

$$\tilde{\alpha}_0^R(x, y) = \frac{2(1+w) - \gamma(x+y) - 2\gamma w(1-y^2)}{4 + 2w(1+y^2) - (\gamma+\mu-\gamma\mu)(x+y)}.$$

Take the partial derivative of the above $\tilde{\alpha}_1^R$ and $\tilde{\alpha}_0^R$ with respect to y, we get:

$$\frac{\partial}{\partial y}\tilde{\alpha}_1^R(x, y) = \frac{2w\left(1+2xy+y^2\right)(1-\gamma)\gamma(1-\mu)}{\left[2w(1-y^2) + (x+y)(\gamma+\mu-\gamma\mu)\right]^2} > 0$$

$$\begin{aligned}\frac{\partial}{\partial y}\tilde{\alpha}_0^R(x, y) \;=\; & \frac{2}{\left[4+2w(1+y^2)-(x+y)(\gamma+\mu-\gamma\mu)\right]^2}\left\{w\left[-4(1+w)y + y(8+8w+2x+y)\gamma - (1+2xy+y^2)\gamma^2\right]\right. \\ & \left. -\gamma + \left[1-\gamma+w(1+\gamma)\left(1-(1+2xy+y^2)\gamma\right)\right]\mu\right\}\end{aligned}$$

Denote the terms in the curly bracket of the above $\frac{\partial}{\partial y}\tilde{\alpha}_0^R$ as N_1. In the next two paragraphs, we will show $\frac{\partial}{\partial y}\tilde{\alpha}_0^R(x, y)\,|_{y=\tilde{y}^R} < 0$ by showing that $N_1\left(y = \tilde{y}^R\right) < 0$ in two cases: (1) when $w > \frac{\mu+\gamma-\gamma\mu}{4}$ and (2) when $w \leq \frac{\mu+\gamma-\gamma\mu}{4}$.

Case (1): If $w \geq \frac{\mu+\gamma-\gamma\mu}{4}$, we must have $\tilde{y}^R = \frac{\mu+\gamma-\gamma\mu}{4w}$. Substitute it into N_1, we get

$$\begin{aligned} N_1\left(y = \tfrac{\mu+\gamma-\gamma\mu}{4w}\right) &= \frac{\gamma(1-\gamma)(1-\mu)\left[-16w^2+(\gamma+\mu-\gamma\mu)^2+8w(-4+x(\gamma+\mu-\gamma\mu))\right]}{16w} \\ &< 0 \end{aligned}$$

because $-16w^2 + (\gamma + \mu - \gamma\mu)^2 < 0$ and $-4 + x(\gamma + \mu - \gamma\mu) < 0$. The last inequality is from $x \leq 1$ and $\gamma + \mu - \gamma\mu < 2$.

Case (2): If $w < \frac{\mu+\gamma-\gamma\mu}{4}$, we must have $\tilde{y}^R = 1$. Substitute it into N_1, we have

$$\begin{aligned} N_1\left(y = 1\right) &= 4w^2(2\gamma-1) + \mu - \gamma(1+\mu) + w\left[-4 + \gamma\left(9 + 2x(1-\gamma)(1-\mu) - 2\gamma(1-\mu) - 3\mu\right) + \mu\right] \\ &= \mu - \gamma(1+\mu) + w\left[4w(2\gamma-1) - 4 + \gamma\left(9 + 2x(1-\gamma)(1-\mu) - 2\gamma(1-\mu) - 3\mu\right) + \mu\right] \\ &< \mu - \gamma - \gamma\mu + w\left[\left(\mu+\gamma-\gamma\mu\right)\left(2\gamma-1\right) - 4 + \gamma\left(9 + 2x(1-\gamma)(1-\mu) - 2\gamma(1-\mu) - 3\mu\right) + \mu\right] \\ &= \mu - \gamma - \gamma\mu + 2w\left[-2 + \gamma\left(4 + x\left(1-\gamma\right)\left(1-\mu\right)\right)\right] \\ &= \mu - \gamma - \gamma\mu + 4w\left(2\gamma-1\right) + 2w\gamma x\left(1-\gamma\right)\left(1-\mu\right) \\ &< \left(\mu + \gamma - \gamma\mu - 2\gamma\right) + \left(\mu+\gamma-\gamma\mu\right)\left(2\gamma-1\right) + 2\gamma\left(1-\gamma\right)\left(1-\mu\right) \\ &= 2\gamma\left(\mu+\gamma-\gamma\mu\right) - 2\gamma + 2\gamma\left(1-\gamma-\mu+\gamma\mu\right) \\ &= 0, \end{aligned}$$

where both inequalities are obtained from substituting $w < \frac{\mu+\gamma-\gamma\mu}{4}$. To summarize both cases (1) and (2), we have proved $\frac{\partial}{\partial y}\tilde{\alpha}_0^R|_{y=\tilde{y}^R} < 0$. Therefore, we have

$$\frac{\partial}{\partial y}\tilde{M}\left(x; y, \gamma\right)|_{y=\tilde{y}^R} = \frac{(\mu+\gamma-\mu\gamma)}{4(1+w)}\left[\frac{\partial}{\partial y}\tilde{\alpha}_1^R|_{y=\tilde{y}^R} - \frac{\partial}{\partial y}\tilde{\alpha}_0^R|_{y=\tilde{y}^R}\right] > 0.$$

Finally, we will show that the first term in (A.9) is positive, i.e., $\frac{\partial}{\partial\gamma}\tilde{M}\left(x; y, \gamma\right) > 0$ for any given $y \in [0,1]$. To see this, note that $\tilde{M}\left(x; y, \gamma\right) = \frac{(\mu+\gamma-\mu\gamma)}{4(1+w)}\left[\tilde{\alpha}_1^R - \tilde{\alpha}_0^R\right]$, where $\frac{(\mu+\gamma-\mu\gamma)}{4(1+w)}$ is positive and always increasing in γ because $\mu < 1$. Therefore, as long as $\frac{\partial}{\partial\gamma}\tilde{\alpha}_1^R - \frac{\partial}{\partial\gamma}\tilde{\alpha}_0^R > 0$, we will have $\frac{\partial}{\partial\gamma}\tilde{M}\left(x; y, \gamma\right) > 0$. Take the partial derivative of $\tilde{\alpha}_1^R$and $\tilde{\alpha}_0^R$ with respect to γ, respectively, we get:

$$\frac{\partial\tilde{\alpha}_1^R}{\partial\gamma} = \frac{\left[x + y + 2w(1-y^2)\right]\left[\mu(x+y) + 2w(1-y^2)\right]}{\left[\left(\mu+\gamma-\gamma\mu\right)\left(x+y\right) + 2w(1-y^2)\right]^2} > 0$$

$$\frac{\partial \tilde{\alpha}_0^R}{\partial \gamma} = \frac{(x+y)\left[\mu(x+y) - 2(1+\mu)\right] - 8w(1-y^2) - 2wy^2(x+y)(1+\mu) - 4w^2(1-y^4)}{\left[4 + 2w(1+y^2) - (\mu+\gamma-\gamma\mu)(x+y)\right]^2} < 0,$$

where the last inequality is because $\mu(x+y) < 2\mu < 2(1+\mu)$. Thus, $\frac{\partial}{\partial\gamma}\tilde{\alpha}_1^R - \frac{\partial}{\partial\gamma}\tilde{\alpha}_0^R > 0$. It follows that $\frac{\partial}{\partial\gamma}\tilde{M}(x; y, \gamma) > 0$ for $\forall y \in [0, 1]$.

Altogether, we have established that all three terms on the left-hand side of (A.9) are positive, so we have $\frac{d}{d\gamma}\tilde{M}\left(x; \tilde{y}^R(\gamma), \gamma\right) > 0$, i.e., $\tilde{M}(x)$ is increasing in γ. □

Proof of Proposition 7:

Proof. **(I) Comparing $\tilde{y}^R$ and y^V (The First Row of Table 1)**

From (A.6), we can verify that $\tilde{y}^R < 1$ for $w > \frac{\sigma_H + \sigma_L\mu}{4(\sigma_H+\sigma_L)}$. Therefore, $y^V = 1 \geq \tilde{y}^R$, where the inequality is strict for $w > \frac{\sigma_H+\sigma_L\mu}{4(\sigma_H+\sigma_L)}$. In addition, we can re-write y^e as:

$$y^e = \begin{cases} \frac{1+\mu}{8w} & \text{if } w > \frac{1+\mu}{8} \\ 1 & \text{otherwise} \end{cases}.$$

Note that $\frac{\sigma_H+\sigma_L\mu}{4(\sigma_H+\sigma_L)} \geq \frac{1+\mu}{8}$ for $\sigma_H \geq \sigma_L$ and $\mu < 1$. Thus, we have

$$\tilde{y}^R - y^e = \begin{cases} \frac{\sigma_H+\sigma_L\mu}{4w(\sigma_H+\sigma_L)} - \frac{1+\mu}{8w} \geq 0 & \text{if } w > \frac{\sigma_H+\sigma_L\mu}{4(\sigma_H+\sigma_L)} \\ 1 - \frac{1+\mu}{8w} > 0 & \text{if } \frac{\sigma_H+\sigma_L\mu}{4(\sigma_H+\sigma_L)} \geq w > \frac{1+\mu}{8} \\ 1 - 1 = 0 & \text{if } w \leq \frac{1+\mu}{8} \end{cases}.$$

The upper branch of $\tilde{y}^R - y^e$ can be simplified to $\frac{(\sigma_H-\sigma_L)(1-\mu)}{8w(\sigma_H+\sigma_L)}$, which is non-negative for $\sigma_H \geq \sigma_L$ and $\mu < 1$. Therefore, we have $\tilde{y}^R \geq y^e$ and the inequality holds when $\sigma_H > \sigma_L$.

(II) Comparing $\tilde{x}^R$ and x^V (The Second Row of Table 1)

We compare $\tilde{x}^R$ and x^V in two special cases: (a) the signal is uninformative, i.e., $\sigma_H = \sigma_L = \sigma$; (b) the signal is informative and symmetric, i.e., $\sigma_H = \gamma$, $\sigma_L = 1-\gamma$ ($\gamma \in \left[\frac{1}{2}, 1\right]$).

(II-a) Uninformative Signal: $\sigma_H = \sigma_L = \sigma$

(i) First, we want to show $\frac{1}{2}\tilde{M}(1) \leq \frac{1}{8}$, so $\tilde{x}^R$ is the unique solution to $\tilde{M}(x) = 2cx$ for $c > \frac{1}{8}$. Based on Lemma A.3, we know that $\tilde{M}(x)$ is decreasing in w and increasing in σ, i.e., $\tilde{M}\left(1|_{w>0,\sigma<1}\right) < \tilde{M}\left(1|_{w=0,\sigma=1}\right)$. When $w = 0$, we have $\tilde{y}^R|_{w=0} = 1$. Recall that when $\tilde{y}^R = 1$,

$\tilde{M}\left(x|\tilde{y}^R=1\right)$ is (A.7). It follows that

$$\tilde{M}\left(x|_{w=0,\sigma=1}\right)=\frac{1-\mu}{2\left[3-\mu-(1+\mu)\,x\right]},$$

from which we get $\tilde{M}\left(1|_{w=0,\sigma=1}\right)=\frac{1}{4}$. Therefore, we have $\frac{1}{2}\tilde{M}(1)\le\frac{1}{8}$ and the inequality is strict for any $w>0$ or $\sigma<1$.

Having shown that $\frac{1}{2}\tilde{M}(1)\le\frac{1}{8}$, we can conclude that for $c>\frac{1}{8}$, $\tilde{x}^R$ is the unique solution to $\tilde{M}\left(x\right)=2cx$. Following the fact that $\tilde{M}(x)$ is increasing in σ (because of Lemma A.3), we know $\tilde{x}^R$ must increase in σ.

(ii) Note that when $\sigma=1$, $\tilde{M}\left(x\right)$ simplifies to $M(x)$ of Proposition 2 for all w. Therefore, we have $\tilde{x}^R|_{\sigma=1}=x^R$. From Proposition 3, we know $x^R<x^V$ for $c>\frac{1}{8}$. Combining with the fact that $\tilde{x}^R$ is increasing in σ, we have $\tilde{x}^R\le\tilde{x}^R|_{\sigma=1}<x^V$ for $c>\frac{1}{8}$.

(II-b) Informative Symmetric Signal: $\sigma_H=\gamma$**,** $\sigma_L=1-\gamma$ $\left(\gamma\in\left[\frac{1}{2},1\right]\right)$

(i) First, we need to show that when $\sigma_H=\gamma\in\left[\frac{1}{2},1\right]$ and $\sigma_L=1-\gamma$, we have $\frac{1}{2}\tilde{M}(1)\le\frac{1}{8}$, so that for $c>\frac{1}{8}$, $\tilde{x}^R$ is the unique solution to $\tilde{M}(x)=2cx$. Because $\tilde{M}(x)$ is increasing in γ (Lemma A.5), we know that $\tilde{M}\left(1|_{\gamma<1}\right)\le\tilde{M}\left(1|_{\gamma=1}\right)$. When $\gamma=1$, it is easy to verify that $\tilde{y}^R|_{\gamma=1}=\min\left\{\frac{1}{4w},\ 1\right\}$ and

$$\tilde{M}\left(x|_{\gamma=1}\right)=\begin{cases}\tilde{M}\left(x|\tilde{y}^R=\frac{1}{4w},\gamma=1\right)=\frac{4w}{16w^2+32w-1-8wx} & \text{if } w>\frac{1}{4}\\ \tilde{M}\left(x|\tilde{y}^R=1,\gamma=1\right)=\frac{1}{2(3+4w-x)} & \text{if } 0\le w\le\frac{1}{4}\end{cases}.$$

Thus, when $w>\frac{1}{4}$, we have $\tilde{M}\left(1|_{\gamma=1}\right)=\frac{4w}{16w^2+24w-1}=\frac{1}{4w+6-\frac{1}{4w}}<\frac{1}{1+6-1}<\frac{1}{4}$, where the first inequality comes from $4w>1$. When $w\le\frac{1}{4}$, we have $\tilde{M}\left(1|_{\gamma=1}\right)=\frac{1}{4(1+2w)}\le\frac{1}{4}$, where the last inequality is true for all $w\ge0$. Therefore, we have $\frac{1}{2}\tilde{M}(1)\le\frac{1}{8}$ and the equality holds only for $\gamma=1$ and $w=0$. It follows that for any $c>\frac{1}{8}\ge\frac{1}{2}\tilde{M}(1)$, $\tilde{x}^R$ is the unique solution to $\tilde{M}(x)=2cx$. Combining with the fact that $\tilde{M}(x)$ is increasing in γ, we know that $\tilde{x}^R$ must be increasing in γ.

(ii) When $\gamma=1$, we can solve $\tilde{x}^R$ by letting $\tilde{M}\left(x|_{\gamma=1}\right)=2cx$. Therefore, for $c>\frac{1}{8}$, we have

$$\tilde{x}^R|_{\gamma=1}=\begin{cases}2+w-\frac{1}{16w}-\sqrt{\left(2+w-\frac{1}{16w}\right)^2-\frac{1}{4c}} & \text{if } w>\frac{1}{4}\\ 2w+\frac{3}{2}-\sqrt{\left(2w+\frac{3}{2}\right)^2-\frac{1}{4c}} & \text{if } 0\le w\le\frac{1}{4}\end{cases}. \tag{A.10}$$

Comparing this $\tilde{x}^R|_{\gamma=1}$ to x^V of Proposition 1, we can show that $\tilde{x}^R|_{\gamma=1}<x^V$ when $\mu<\tilde{\mu}\left(c,w\right)$;

and $\tilde{x}^R|_{\gamma=1} \geq x^V$ when $\mu \geq \tilde{\mu}(c,w)$, where

$$\tilde{\mu}(c,w) = \begin{cases} \frac{c\left(1-8w-16w^2\right)}{8cw-\sqrt{-64cw^2+c^2[1-16w(2+w)]^2}} & \text{if } w > \frac{1}{4} \\ \frac{4w\left[c+\sqrt{c^2(3+4w)^2-c}\right]}{8c(1+w)(1+2w)-1} & \text{if } 0 \leq w \leq \frac{1}{4}. \end{cases} \tag{A.11}$$

In addition, we know that $\tilde{x}^R$ is increasing in γ. Consequently, when $\mu < \tilde{\mu}(c,w)$, we must have $\tilde{x}^R|_{\frac{1}{2}\leq\gamma\leq 1} \leq \tilde{x}^R|_{\gamma=1} < x^V$. When $\mu \geq \tilde{\mu}(c,w)$, we need to compare $\tilde{x}^R|_{\gamma=\frac{1}{2}}$ to x^V. Clearly, $\gamma = \frac{1}{2}$ implies $\sigma_H = \sigma_L = \frac{1}{2} = \sigma$. According to Part (II-a), we know that $\tilde{x}^R|_{\gamma=\sigma=\frac{1}{2}} < x^V$. Combining with the fact that $\tilde{x}^R$ is increasing in γ, we know there exists a threshold value $\tilde{\gamma}(\mu,c,w)$, such that $\tilde{x}^R < x^V$ for $\gamma \in [\frac{1}{2}, \tilde{\gamma})$ and $\tilde{x}^R \geq x^V$ for $\gamma \in [\tilde{\gamma}, 1]$.

(III) Comparing $EV\left(\tilde{x}^R, \tilde{y}^R\right)$ and $EV\left(x^V, y^V\right)$ (The Third Row of Table 1)

Recall that $EV(x,y) = \frac{1}{4}(1+\mu)(x+y)$. To compare $EV\left(\tilde{x}^R, \tilde{y}^R\right)$ and $EV\left(x^V, y^V\right)$, we compare $\left(\tilde{x}^R + \tilde{y}^R\right)$ and $\left(x^V + y^V\right)$ in two special cases: (a) the signal is uninformative, i.e., $\sigma_H = \sigma_L = \sigma$; (b) the signal is informative and symmetric, i.e., $\sigma_H = \gamma$, $\sigma_L = 1-\gamma$ ($\gamma \in \left[\frac{1}{2}, 1\right]$).

(III-a) Uninformative Signal: $\sigma_H = \sigma_L = \sigma$

In this case, because we have $\tilde{x}^R < x^V$ and $\tilde{y}^R < y^V$ for $\forall\sigma \in (0,1]$, it follows that $EV\left(\tilde{x}^R, \tilde{y}^R\right) < EV\left(x^V, y^V\right)$ for all σ.

(III-b) Informative Symmetric Signal: $\sigma_H = \gamma$, $\sigma_L = 1-\gamma$ ($\gamma \in \left[\frac{1}{2}, 1\right]$)

Because $\tilde{x}^R$ and $\tilde{y}^R$ are increasing in γ, and x^V and y^V are independent of γ, $EV\left(\tilde{x}^R, \tilde{y}^R\right) > EV\left(x^V, y^V\right)$ holds for some $\gamma \in \left(\frac{1}{2}, 1\right)$ only if $EV\left(\tilde{x}^R, \tilde{y}^R\right)|_{\gamma=1} > EV\left(x^V, y^V\right)$. Therefore, we next examine the case of $\gamma = 1$ and show under what conditions $EV\left(\tilde{x}^R, \tilde{y}^R\right)|_{\gamma=1} > EV\left(x^V, y^V\right)$.

First, if $w \leq \frac{1}{4}$ when $\gamma = 1$, we have $\tilde{y}^R = 1 = y^V$, thus $EV\left(\tilde{x}^R, \tilde{y}^R\right) > EV\left(x^V, y^V\right)$ if and only if $\tilde{x}^R > x^V$, which holds when $\mu > \frac{4w\left[c+\sqrt{c^2(3+4w)^2-c}\right]}{8c(1+w)(1+2w)-1}$ according to the proof of Part (II-b) (Equation (A.11)). Therefore, $\hat{\mu}(c,w) = \frac{4w\left[c+\sqrt{c^2(3+4w)^2-c}\right]}{8c(1+w)(1+2w)-1}$ for $\forall w \in (0, \frac{1}{4}]$, $c \in \left(\frac{1}{8}, 1\right)$.

Second, if $w > \frac{1}{4}$ when $\gamma = 1$, we have $\tilde{y}^R < y^V$. Let $f(w,c) \equiv \tilde{x}^R + \tilde{y}^R = [2 + w - \frac{1}{16w} - \sqrt{\left(2+w-\frac{1}{16w}\right)^2 - \frac{1}{4c}}] + \frac{1}{4w}$, and let $g(\mu,c) \equiv x^V + y^V = \frac{c(3-\mu)-\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)} + 1$. $EV\left(\tilde{x}^R, \tilde{y}^R\right) > EV\left(x^V, y^V\right)$ if and only if $f(w,c) > g(\mu,c)$. As we can show, $f(w,c)$ is decreasing in w, and $g(\mu,c)$ is decreasing in μ. As we can verify, $\forall c \in \left(\frac{1}{8}, 1\right)$, $g(\mu=1,c) < f\left(w=\frac{1}{4}, c\right) < g(\mu=0,c)$ and $f(w=1,c) < g(\mu=1,c) < g(\mu=0,c)$. Therefore, there exists a $\hat{w}(c) \in \left(\frac{1}{4}, 1\right)$ such that $f(\hat{w},c) = g(\mu=1,c)$. As a result, when $\frac{1}{4} < w < \hat{w}$, there exists a $\hat{\mu}(c,w) \in (0,1)$ such that $g(\hat{\mu}, c) = f(w, c)$ and $g(\mu, c) < f(w, c)$ if $\mu > \hat{\mu}$.

Altogether, we have shown that for any $c \in \left(\frac{1}{8}, 1\right)$, $w \in (0, \hat{w}(c))$, and $\mu \in (\hat{\mu}(c, w), 1)$, $EV\left(\tilde{x}^R, \tilde{y}^R\right)|_{\gamma=1} > EV\left(x^V, y^V\right)$. Because $EV\left(\tilde{x}^R, \tilde{y}^R\right)|_{\gamma=\frac{1}{2}} < EV\left(x^V, y^V\right)$ always holds, there exists a threshold $\hat{\gamma}(c, w, \mu) \in \left(\frac{1}{2}, 1\right)$ such that when $\hat{\gamma} < \gamma \leq 1$, $EV\left(\tilde{x}^R, \tilde{y}^R\right) > EV\left(x^V, y^V\right)$. $\square$

Proof of Proposition 8:

Proof. As we can show, when V alone is observed as a signal, the equilibrium effort

$$\hat{y}^V = \begin{cases} 1 & \text{if } \frac{1+\mu}{8w} \geq 1 \\ \frac{1+\mu}{8w} & \text{if } \lambda < \frac{1+\mu}{8w} < 1 \\ \lambda & \text{if } \frac{1+\mu}{8w} \leq \lambda, \end{cases} \tag{A.12}$$

and $\hat{x}^V$ is the unique solution within $[0, 1]$ to the equation $K\left(\hat{x}^V, \hat{y}^V\right) = 2c\hat{x}^V$ (for $c > \frac{1}{8}$), where

$$K(x, y) = \left[\alpha_1^V(x, y) - \alpha_0^V(x, y)\right] \frac{1+\mu}{4} = \frac{1-\mu}{2\left[4 - (1+\mu)(x+y)\right]}. \tag{A.13}$$

When R alone is observed as a signal, the equilibrium effort

$$\hat{y}^R = \begin{cases} 1 & \text{if } \frac{1+\mu}{8w} \geq 1 \\ \frac{1+\mu}{8w} & \text{if } \frac{1+\mu}{8w} < 1, \end{cases} \tag{A.14}$$

and $\hat{x}^R$ is the unique solution within $[0, 1]$ to the equation $M\left(\hat{x}^R, \hat{y}^R\right) = 2c\hat{x}^R$ (for $c > \frac{1}{8}$), where

$$\begin{aligned} M(x, y) &= \left[\alpha_1^R(x, y) - \alpha_0^R(x, y)\right] \frac{(1+\mu)}{4(1+w)} \\ &= \left[\frac{(x+y) + 2w(1-y^2)}{(1+\mu)(x+y) + 4w(1-y^2)} - \frac{2(1+w) - (x+y) - 2w(1-y^2)}{4(1+w) - (1+\mu)(x+y) - 4w(1-y^2)}\right] \frac{(1+\mu)}{4(1+w)}. \end{aligned} \tag{A.15}$$

(i) Comparing (A.12) and (A.14), it is easy to see that $\hat{y}^R = \hat{y}^V$ if $\frac{1+\mu}{8w} \geq \lambda$, and $\hat{y}^R < \hat{y}^V$ if $\frac{1+\mu}{8w} < \lambda$.

(ii) To compare $\hat{x}^V$ and $\hat{x}^R$, notice that $M(x, y) - K(x, y)$ can be simplified as

$$\frac{1-\mu}{2} \cdot \frac{-4w\left(1-y^2\right)\left[4 - (1+\mu)(x+y)\right] - 4wy^2\left[4w\left(1-y^2\right) + (1+\mu)(x+y)\right]}{\left[4 - (1+\mu)(x+y) + 4wy^2\right]\left[4w\left(1-y^2\right) + (1+\mu)(x+y)\right]\left[4 - (1+\mu)(x+y)\right]} < 0.$$

Therefore, $M(x,y) < K(x,y)$ for any x, y. If $\frac{1+\mu}{8w} \geq \lambda$, $\hat{y}^R = \hat{y}^V$. Therefore, $M\left(x, \hat{y}^R\right) < K\left(x, \hat{y}^V\right)$ for any x. As a result, $\hat{x}^R < \hat{x}^V$ for $\lambda \leq \frac{1+\mu}{8w}$.

If $\frac{1+\mu}{8w} < \lambda$, substitute $\hat{y}^V = \lambda$ and $\hat{y}^R = \frac{1+\mu}{8w}$ into $K(x,y)$ and $M(x,y)$, respectively. We have

$$\begin{aligned} & M\left(x, \hat{y}^R | w = \frac{1+\mu}{8\lambda}\right) - K\left(x, \hat{y}^V\right) \\ = & -\frac{(1-\mu)}{2}\left[\frac{1}{4-(1+\mu)(x+\lambda)} - \frac{4\lambda(x+\lambda)}{1+2x\lambda+\lambda^2} \cdot \frac{1}{2\left[4-(1+\mu)(x+\lambda)\right]+\lambda(1+\mu)}\right] \\ < & -\frac{(1-\mu)}{2}\left[\frac{1}{4-(1+\mu)(x+\lambda)} - 2 \cdot \frac{1}{2\left[4-(1+\mu)(x+\lambda)\right]}\right] = 0. \end{aligned}$$

We can further show that $M\left(x, \hat{y}^R\right)$ is decreasing in w. Notice that $K\left(x, \hat{y}^V\right)$ is independent of w. Therefore, for $\forall w > \frac{1+\mu}{8\lambda}$, $M\left(x, \hat{y}^R\right) < M(x, \hat{y}^R | w = \frac{1+\mu}{8\lambda}) < K(x, \hat{y}^V)$ for any x. As a result, $\hat{x}^R < \hat{x}^V$ for $\lambda > \frac{1+\mu}{8w}$. □

Proof of Proposition 9:

Proof. In the scenario when the client incorporates symmetric private signal into the review (with $\sigma_H = \gamma$ and $\sigma_L = 1-\gamma$, $\gamma \in \left[\frac{1}{2}, 1\right]$), when V alone is observed as a signal, the equilibrium effort remains the same as in the baseline model, that is,

$$\hat{y}^V = \begin{cases} 1 & \text{if } \frac{1+\mu}{8w} \geq 1 \\ \frac{1+\mu}{8w} & \text{if } \lambda < \frac{1+\mu}{8w} < 1 \\ \lambda & \text{if } \frac{1+\mu}{8w} \leq \lambda, \end{cases} \tag{A.16}$$

and $\hat{x}^V$ is the unique solution within $[0,1]$ to the equation $K\left(x, \hat{y}^V\right) = 2cx$ (for $c > \frac{1}{8}$), where

$$K(x,y) = \left[\alpha_1^V(x,y) - \alpha_0^V(x,y)\right]\frac{1+\mu}{4} = \frac{1-\mu}{2\left[4-(1+\mu)(x+y)\right]}. \tag{A.17}$$

When R alone is observed as a signal, as we can show, the equilibrium effort can be derived as

$$\hat{\hat{y}}^R = \begin{cases} 1 & \text{if } w \leq \frac{1+\mu}{8} \\ \frac{1+\mu}{8w} & \text{if } \frac{1+\mu}{8} < w \leq \frac{1+\mu}{8\lambda} \\ \lambda & \text{if } \frac{1+\mu}{8\lambda} < w \leq \frac{\gamma+(1-\gamma)\mu}{4\lambda} \\ \frac{\gamma+(1-\gamma)\mu}{4w} & \text{if } w > \frac{\gamma+(1-\gamma)\mu}{4\lambda} \end{cases} \tag{A.18}$$

and $\hat{\hat{x}}^R$ is the unique solution within $[0,1]$ to the equation $\tilde{M}\left(x, \hat{\hat{y}}^R\right) = 2cx$ (for $c > \frac{1}{8}$), where

$$\begin{aligned} \tilde{M}(x,y) &= \left[\tilde{\alpha}_1^R(x,y) - \tilde{\alpha}_0^R(x,y)\right] \frac{\gamma + (1-\gamma)\mu}{4(1+w)} \\ &= \left[\frac{\gamma(x+y) + 2\gamma w(1-y^2)}{(\gamma + (1-\gamma)\mu)(x+y) + 2w(1-y^2)} - \frac{2(1+w) - \gamma(x+y) - 2\gamma w(1-y^2)}{4(1+w) - (\gamma + (1-\gamma)\mu)(x+y) - 2w(1-y^2)}\right] \frac{\gamma + (1-\gamma)\mu}{4(1+w)} \end{aligned} \tag{A.19}$$

(i) Comparing (A.16) and (A.18), it is easy to conclude that $\hat{\hat{y}}^R = \hat{y}^V$ if $w \leq \frac{\gamma+(1-\gamma)\mu}{4\lambda}$, and $\hat{\hat{y}}^R < \hat{y}^V$ if $w > \frac{\gamma+(1-\gamma)\mu}{4\lambda}$.

(ii) To compare $\hat{\hat{x}}^R$ and $\hat{x}^V$ as γ varies, we first show that $\tilde{M}\left(x, \hat{\hat{y}}^R\right)$ is increasing in γ. To show this, note that $\frac{d}{d\gamma}\tilde{M}\left(x, \hat{\hat{y}}^R\right) = \frac{\partial}{\partial\gamma}\tilde{M}\left(x, \hat{\hat{y}}^R\right) + \frac{\partial}{\partial y}\tilde{M}\left(x, \hat{\hat{y}}^R\right)\frac{d}{d\gamma}\hat{\hat{y}}^R$. As we can check, $\frac{\partial}{\partial\gamma}\tilde{\alpha}_1^R(x,y) > 0$ and $\frac{\partial}{\partial\gamma}\tilde{\alpha}_0^R(x,y) < 0$ for any x and y. Therefore, we can easily conclude that $\frac{\partial}{\partial\gamma}\tilde{M}(x,y) = \left[\frac{\partial}{\partial\gamma}\tilde{\alpha}_1^R - \frac{\partial}{\partial\gamma}\tilde{\alpha}_0^R\right]\frac{\gamma+(1-\gamma)\mu}{4(1+w)} + \left(\tilde{\alpha}_1^R - \tilde{\alpha}_0^R\right)\frac{1-\mu}{4(1+w)} > 0$ for any x and y (notice that $\tilde{\alpha}_1^R > \tilde{\alpha}_0^R$ always holds). For $w \leq \frac{\gamma+(1-\gamma)\mu}{4\lambda}$, $\hat{\hat{y}}^R$ is independent of γ, so $\frac{d}{d\gamma}\hat{\hat{y}}^R = 0$ in this case. As a result, $\frac{d}{d\gamma}\tilde{M}\left(x, \hat{\hat{y}}^R\right) > 0$. For $w > \frac{\gamma+(1-\gamma)\mu}{4\lambda}$, $\frac{d}{d\gamma}\hat{\hat{y}}^R = \frac{1-\mu}{4\lambda} > 0$. We thus need to check the sign of $\frac{\partial}{\partial y}\tilde{M}\left(x, \hat{\hat{y}}^R\right)$. As we can check, $\frac{\partial}{\partial y}\tilde{\alpha}_1^R(x,y) > 0$ for any x and y, and $\frac{\partial}{\partial y}\tilde{\alpha}_0^R(x, \hat{\hat{y}}^R) < 0$ after substituting $\hat{\hat{y}}^R = \frac{\gamma+(1-\gamma)\mu}{4w}$ into the expression of $\frac{\partial}{\partial y}\tilde{\alpha}_0^R$. Therefore, $\frac{\partial}{\partial y}\tilde{M}\left(x, \hat{\hat{y}}^R\right) > 0$, and hence $\frac{d}{d\gamma}\tilde{M}\left(x, \hat{\hat{y}}^R\right) > 0$. As a result, $\tilde{M}\left(x, \hat{\hat{y}}^R\right)$ is increasing in γ. Therefore, we can conclude that $\hat{\hat{x}}^R$, as the solution to $\tilde{M}\left(x, \hat{\hat{y}}^R\right) = 2cx$, is also increasing in γ.

We next check the extreme case when $\gamma = 1$, and compare $\hat{\hat{x}}^R$ and $\hat{x}^V$ in this case. When $\gamma = 1$, $\tilde{M}(x,y)$ in (A.19) simplifies to $\tilde{M}(x,y|\gamma=1) = \frac{1}{2(4-x-y)+4w(1+y^2)}$. As a result, we can explicitly solve $\tilde{M}(x,y) = 2cx$ for $\hat{\hat{x}}^R(y)$ such that

$$\hat{\hat{x}}^R(y)\,|_{\gamma=1} = \frac{c\left[4 - y + 2w\left(1+y^2\right)\right] - \sqrt{c\left[c\left(4 - y + 2w\left(1+y^2\right)\right)^2 - 1\right]}}{2c}. \tag{A.20}$$

Similarly, we can explicitly solve $K(x,y) = 2cx$ for $\hat{x}^V(y)$ such that

$$\hat{x}^V(y) = \frac{c(4-(1+\mu)y) - \sqrt{c^2(4-(1+\mu)y)^2 - c(1-\mu^2)}}{2c(1+\mu)}. \tag{A.21}$$

Notice that given y, $\hat{\tilde{x}}^R(y)|_{\gamma=1}$ is independent of μ, whereas $\hat{x}^V(y)$ is decreasing in μ given y (because $\frac{\partial}{\partial\mu}K(x,y) = \frac{-(2-x-y)}{[4-(1+\mu)(x+y)]^2} < 0$).

Substituting $\hat{\tilde{y}}^R$ from (A.18) into (A.20) and $\hat{y}^V$ from (A.16) into (A.21), we can compare $\hat{\tilde{x}}^R$ and $\hat{x}^V$ within different regions:

(a) When $w \leq \frac{1+\mu}{8}$, substituting in $\hat{\tilde{y}}^R = 1$ and $\hat{y}^V = 1$, and solving $\hat{\tilde{x}}^R = \hat{x}^V$, we have

$$\tilde{\mu}_1 = \frac{4w\left[c + \sqrt{c^2(3+4w)^2 - c}\right]}{8c(1+w)(1+2w) - 1}. \tag{A.22}$$

Because $\hat{\tilde{x}}^R$ is independent of μ, and $\hat{x}^V$ is decreasing in μ, $\hat{\tilde{x}}^R > \hat{x}^V$ if $\mu > \tilde{\mu}_1$.

(b) When $\frac{1+\mu}{8} < w \leq \frac{1+\mu}{8\lambda}$, $\hat{\tilde{y}}^R = \hat{y}^V = \frac{1+\mu}{8w}$. Notice that because both $\hat{\tilde{y}}^R$ and $\hat{y}^V$ depend on μ, we cannot obtain an explicit sufficient and necessary condition on μ for $\hat{\tilde{x}}^R\left(\hat{\tilde{y}}^R\right) > \hat{x}^V\left(\hat{y}^V\right)$ to hold. Nevertheless, it is possible to obtain a sufficient condition instead. Notice that $\hat{x}^V\left(\hat{y}^V = \frac{1+\mu}{8w}\right) < \hat{x}^V(y=1)$ (because $\hat{x}^V(y)$ is increasing in y), and $\hat{\tilde{x}}^R\left(\hat{\tilde{y}}^R = \frac{1+\mu}{8w}\right) > \hat{\tilde{x}}^R(y=\lambda)$ (because $\hat{\tilde{x}}^R(y)$ is increasing in y for $y < \frac{1}{4w}$, and $\frac{1}{4w} > \frac{1+\mu}{8w} > \lambda$ in the region we are discussing). Therefore, if $\hat{\tilde{x}}^R(y=\lambda) > \hat{x}^V(y=1)$ holds, then $\hat{\tilde{x}}^R\left(\hat{\tilde{y}}^R = \frac{1+\mu}{8w}\right) > \hat{x}^V\left(\hat{y}^V = \frac{1+\mu}{8w}\right)$ holds as well. Solving $\hat{\tilde{x}}^R(y=\lambda) = \hat{x}^V(y=1)$ for μ, we have

$$\tilde{\mu}_2 = \frac{\left(1-\lambda+2w\left(1+\lambda^2\right)\right)\left[c + \sqrt{c^2\left(4-\lambda+2w\left(1+\lambda^2\right)\right)^2 - c}\right]}{c\left(3-\lambda+2w\left(1+\lambda^2\right)\right)\left(5-\lambda+2w\left(1+\lambda^2\right)\right) - 1}. \tag{A.23}$$

Because $\hat{\tilde{x}}^R(y=\lambda)$ is independent of μ, and $\hat{x}^V(y=1)$ is decreasing in μ, $\hat{\tilde{x}}^R(y=\lambda) > \hat{x}^V(y=1)$ if $\mu > \tilde{\mu}_2$. As a result, we have a sufficient condition: if $\mu > \tilde{\mu}_2$, $\hat{\tilde{x}}^R\left(\hat{\tilde{y}}^R\right) > \hat{x}^V\left(\hat{y}^V\right)$.

(c) When $\frac{1+\mu}{8\lambda} < w \leq \frac{1}{4\lambda}$, $\hat{\tilde{y}}^R = \lambda$ and $\hat{y}^V = \lambda$. Substitute them into (A.20) and (A.21), and solving $\hat{\tilde{x}}^R = \hat{x}^V$ for μ, we have

$$\tilde{\mu}_3 = \frac{2\left(1+\lambda^2\right)w\left[\lambda c + \sqrt{c^2\left(4-\lambda+2w\left(1+\lambda^2\right)\right)^2 - c}\right]}{4c(2+w+\lambda^2 w)(2+w+\lambda^2 w - \lambda) - 1}. \tag{A.24}$$

Again, because $\hat{\hat{x}}^R$ is independent of μ, and $\hat{x}^V$ is decreasing in μ, $\hat{\hat{x}}^R > \hat{x}^V$ if $\mu > \tilde{\mu}_3$.

(d) When $w > \frac{1}{4\lambda}$, $\hat{\hat{y}}^R = \frac{1}{4w}$ and $\hat{y}^V = \lambda$. Substitute them into (A.20) and (A.21), and solving $\hat{\hat{x}}^R = \hat{x}^V$ for μ, we have

$$\tilde{\mu}_4 = \frac{c\left(1 - 8\lambda w - 16w^2\right)}{8c\lambda w - \sqrt{-64cw^2 + c^2\left[1 - 16w(2+w)\right]^2}}. \tag{A.25}$$

Similarly, we have $\hat{\hat{x}}^R > \hat{x}^V$ if $\mu > \tilde{\mu}_4$.

Combining (a) through (d), we characterize the regions in which $\hat{\hat{x}}^R > \hat{x}^V$ holds, under the extreme case of $\gamma = 1$. Figure 6 illustrates such regions (i.e., regions (a) and (b), as regions (c) and (d) not applicable) when $\lambda = 0$ (i.e., the client fully determines his own effort).

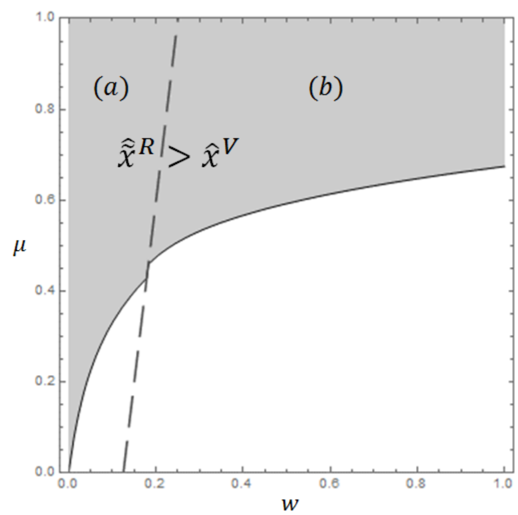


Figure 6: Comparison of Equilibrium Efforts ($\gamma = 1$, $\lambda = 0$, $c = 0.15$)

In the other extreme case when $\gamma = \frac{1}{2}$, as we can show, $\hat{\hat{x}}^R\left(\sigma_H = \sigma_L = \frac{1}{2}\right) < \hat{\hat{x}}^R\left(\sigma_H = \sigma_L = 1\right) < \hat{x}^V$ (Notice that when $\sigma_H = \sigma_L$, $\hat{\hat{y}}^R = y^R$, so the results from the original model apply).

Altogether, we have shown that $\hat{\hat{x}}^R$ is increasing in γ, whereas $\hat{x}^V$ is independent of γ; when $\gamma = \frac{1}{2}$, $\hat{\hat{x}}^R < \hat{x}^V$; when $\gamma = 1$, there exists (c, w, μ, λ) such that $\hat{\hat{x}}^R > \hat{x}^V$ holds. Therefore, we can conclude that there exists (c, w, μ, λ) such that for a certain threshold $\tilde{\gamma}(c, w, \mu, \lambda) \in \left(\frac{1}{2}, 1\right)$, $\hat{\hat{x}}^R \leq \hat{x}^V$ when $\gamma \leq \tilde{\gamma}$, and $\hat{\hat{x}}^R > \hat{x}^V$ when $\gamma > \tilde{\gamma}$. $\square$

S2 Different Distributions for ξ and ε

S2.1 Arbitrary Support

In this section, we extend the supports of ξ and ε to show that our main results do not depend on the specific choices of support in the paper. We now let ξ follow a uniform distribution over an arbitrary support $[-b_1, a_1]$, and ε follow a uniform distribution over $[-b_2, a_2]$. Thus, we have the cdf's of the two random variables as $G(\xi) = \frac{\xi+b_1}{a_1+b_1}$ and $F(\varepsilon) = \frac{\varepsilon+b_2}{a_2+b_2}$. Notice that the original model in the paper corresponds to $a_1 = 0$, $b_1 = 2$, $a_2 = w$, and $b_2 = 1$.

Solve for x^V and y^V: Substituting $G(\xi)$ into (2), we have $\Pr\{V = 1|\theta\} = \frac{\mu_\theta(x+y)+a_1}{a_1+b_1}$. Therefore, the expected payoff of the service provider can be derived as

$$
\begin{aligned}
E\pi &= \alpha_0^V + \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left[\Pr\left(V = 1|\theta = H\right) + \Pr\left(V = 1|\theta = L\right)\right] - cx^2 \\
&= \alpha_0^V + \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\frac{(1+\mu)(x+y) + 2a_1}{a_1 + b_1} - cx^2,
\end{aligned}
$$

where $\alpha_1^V = \frac{a_1+x^V+y^V}{2a_1+(1+\mu)(x^V+y^V)}$ and $\alpha_0^V = \frac{b_1-x^V-y^V}{2b_1-(1+\mu)(x^V+y^V)}$. Taking the first derivative with respect to x and y, we have

$$
\begin{aligned}
\frac{\partial E\pi}{\partial x} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\frac{1+\mu}{a_1+b_1} - 2cx \\
\frac{\partial E\pi}{\partial y} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\frac{1+\mu}{a_1+b_1}
\end{aligned}
$$

Because $\frac{\partial E\pi}{\partial y}$ is a positive constant, we have $y^V = 1$. Meanwhile, x^V is the solution within $[0, 1]$ to $K\left(x, y^V\right) = 2cx$, where $K(x, y) = \frac{(1+\mu)}{2(b_1+a_1)}\left[\alpha_1^V(x, y) - \alpha_0^V(x, y)\right]$.

Solve for x^R and y^R: Substituting $G(\xi)$ and $F(\varepsilon)$ into (10), we have

$$
\begin{aligned}
\Pr\{R = 1|\theta\} &= 1 - F\left(wy^2\right) + \left[F\left(wy^2\right) - F\left(wy^2 - 1\right)\right]\left[1 - G\left(-\mu_\theta(x+y)\right)\right] \\
&= \frac{a_2 - wy^2}{a_2 + b_2} + \frac{\mu_\theta(x+y) + a_1}{(b_1+a_1)(b_2+a_2)}
\end{aligned}
$$

Therefore, the expected payoff of the service provider can be derived as

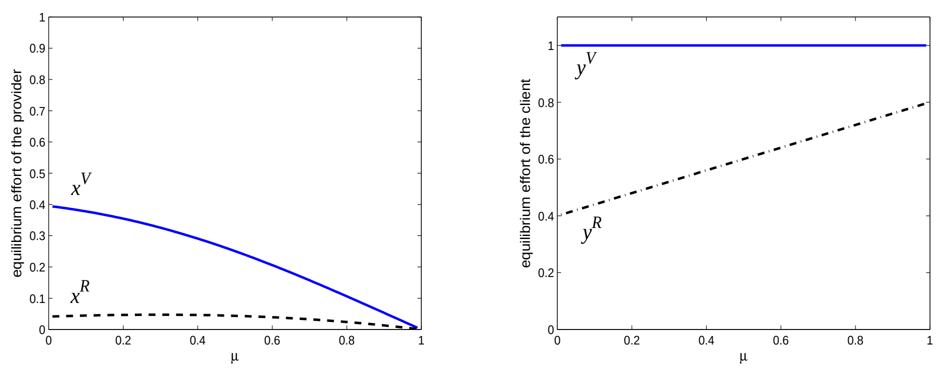
$$
\begin{aligned}
E\pi &= \alpha_0^R + \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\left[\Pr\left(R=1|\theta=H\right) + \Pr\left(R=1|\theta=L\right)\right] - cx^2 \\
&= \alpha_0^R + \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\left[\frac{2\left(a_2 - wy^2\right)}{a_2+b_2} + \frac{\left(1+\mu\right)\left(x+y\right)+2a_1}{(a_1+b_1)(a_2+b_2)}\right] - cx^2,
\end{aligned}
$$

where $\alpha_1^R = \frac{a_1+(x^R+y^R)+(b_1+a_1)\left(a_2-w\left(y^R\right)^2\right)}{2a_1+(1+\mu)(x^R+y^R)+2(b_1+a_1)\left(a_2-w(y^R)^2\right)}$ and $\alpha_0^R = \frac{(b_1+a_1)\left(b_2+w\left(y^R\right)^2\right)-a_1-(x^R+y^R)}{2(b_1+a_1)\left(b_2+w(y^R)^2\right)-2a_1-(1+\mu)(x^R+y^R)}$. Taking the first derivative with respect to x and y, we have

$$
\begin{aligned}
\frac{\partial E\pi}{\partial x} &= \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\frac{1+\mu}{(a_1+b_1)(a_2+b_2)} - 2cx \\
\frac{\partial E\pi}{\partial y} &= \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\frac{1}{a_2+b_2}\left(-4wy + \frac{1+\mu}{a_1+b_1}\right)
\end{aligned}
$$

As a result, we have $y^R = \min\left\{\frac{(1+\mu)}{4w(a_1+b_1)},\ 1\right\}$, and x^R is the solution within $[0,1]$ to $M\left(x, y^R\right) = 2cx$, where $M\left(x,y\right) = \frac{(1+\mu)}{2(a_1+b_1)(a_2+b_2)}\left[\alpha_1^R\left(x,y\right) - \alpha_0^R\left(x,y\right)\right]$.

Compare x^V to x^R, and y^V to y^R: It is easy to see that $y^R \le y^V$ always holds. We numerically solve and compare x^V and x^R for different values of a_1, b_1, a_2 and b_2. We find that the general result that $x^R < x^V$ continues to hold. Figure 7 illustrates a representative comparison result.



(a) x^V vs. x^R (b) y^V vs. y^R

Figure 7: Equilibrium Comparison ($\xi \sim U\left[-2.25, 0.25\right]$, $\varepsilon \sim U\left[-1.25, 0.5\right]$, $c = 0.15$, $w = 0.25$)

S2.2 Beta Distribution

In this section, we extend the distribution of ξ and ε from uniform distribution to a more general distribution family, Beta distribution. Beta distribution is the most commonly used and well studied distribution family over finite supports. Specifically, we let z_1 and z_2 be random variables following Beta distribution over $[0,1]$ with cdf $L(z_i) = 1-(1-z_i)^{\beta_i}$, where $\beta_i > 0$, $z_i \in [0,1]$, and $i = 1,2$. When $\beta_i = 1$, the distribution reduces to uniform distribution; when $\beta_i > 1$ ($\beta_i < 1$), the distribution skews to the right (the left). We then rescale z_1 and z_2 to fit the supports of ξ and ε such that $\xi = 2(z_1 - 1) \in [-2,0]$ and $\varepsilon = z_2(1+w) - 1 \in [-1, w]$. Consequently, the cdf's of ξ and ε can be derived as $G(\xi) = L\left(\frac{\xi}{2}+1\right) = 1 - \left(-\frac{\xi}{2}\right)^{\beta_1}$ and $F(\varepsilon) = L\left(\frac{\varepsilon+1}{1+w}\right) = 1 - \left(1 - \frac{\varepsilon+1}{1+w}\right)^{\beta_2}$.

Solve for x^V and y^V: Substituting $G(\xi)$ into (2), we have $\Pr\{V=1|\theta\} = \left[\frac{\mu_\theta(x+y)}{2}\right]^{\beta_1}$. Therefore, the expected payoff of the service provider can be derived as

$$E\pi = \alpha_0^V + \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left(1+\mu^{\beta_1}\right)\left(\frac{x+y}{2}\right)^{\beta_1} - cx^2.$$

Taking the first derivative with respect to x and y, we have

$$\begin{aligned}\frac{\partial E\pi}{\partial x} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left(1+\mu^{\beta_1}\right)\frac{\beta_1}{2}\left(\frac{x+y}{2}\right)^{\beta_1-1} - 2cx \\ \frac{\partial E\pi}{\partial y} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left(1+\mu^{\beta_1}\right)\frac{\beta_1}{2}\left(\frac{x+y}{2}\right)^{\beta_1-1}\end{aligned}$$

Because $\frac{\partial E\pi}{\partial y} > 0$ for any y, therefore, $y^V = 1$. Meanwhile, x^V is the solution within $[0,1]$ to $K\left(x, y^V\right) = 2cx$, where $K(x,y) = \frac{\beta_1}{4}\left[\alpha_1^V(x,y) - \alpha_0^V(x,y)\right]\left(1+\mu^{\beta_1}\right)\left(\frac{x+y}{2}\right)^{\beta_1-1}$.

Solve for x^R and y^R: Substituting $G(\xi)$ and $F(\varepsilon)$ into (10), we have

$$\Pr\{R=1|\theta\} = \left[\frac{w(1-y^2)}{1+w}\right]^{\beta_2} + \left[\frac{\mu_\theta(x+y)}{2}\right]^{\beta_1}\left[\left(1-\frac{wy^2}{1+w}\right)^{\beta_2} - \left(\frac{w(1-y^2)}{1+w}\right)^{\beta_2}\right].$$

Therefore, the expected payoff of the service provider can be derived as

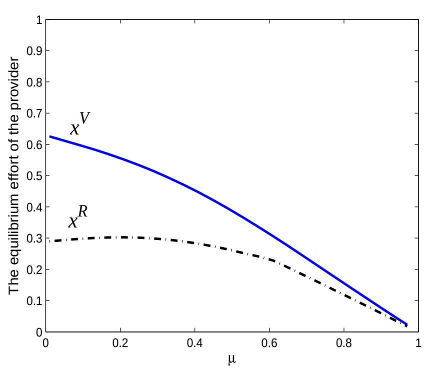
$$
\begin{aligned}
E\pi &= \alpha_0^R + \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\left\{2\left[\frac{w(1-y^2)}{1+w}\right]^{\beta_2}\right. \\
&\left. + \left(1+\mu^{\beta_1}\right)\left(\frac{x+y}{2}\right)^{\beta_1}\left[\left(1-\frac{wy^2}{1+w}\right)^{\beta_2} - \left(\frac{w(1-y^2)}{1+w}\right)^{\beta_2}\right]\right\} - cx^2.
\end{aligned}
$$

Based on the first order condition, $\left(x^R, y^R\right)$ is the solution within $[0,1]$ to the two simultaneous equations: $\frac{\partial E\pi}{\partial x} = 0$ and $\frac{\partial E\pi}{\partial y} = 0$, which can be derived as

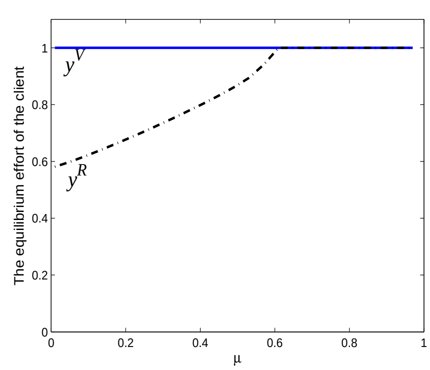
$$
\begin{cases}
\frac{1}{4}\left(\alpha_1^R - \alpha_0^R\right) D_1 \left[\left(1-\frac{wy^2}{1+w}\right)^{\beta_2} - \left(\frac{w(1-y^2)}{1+w}\right)^{\beta_2}\right] = 2cx \\
2D_3 + \frac{D_1}{2}\left[\left(1-\frac{wy^2}{1+w}\right)^{\beta_2} - \left(\frac{w(1-y^2)}{1+w}\right)^{\beta_2}\right] + (D_2 - D_3)\left(1+\mu^{\beta_1}\right)\left(\frac{x+y}{2}\right)^{\beta_1} = 0,
\end{cases}
$$

where $D_1 = \left(1+\mu^{\beta_1}\right)\beta_1\left(\frac{x+y}{2}\right)^{\beta_1 - 1}$, $D_2 = \beta_2\left(1-\frac{wy^2}{1+w}\right)^{\beta_2-1}\left(\frac{-2wy}{1+w}\right)$ and $D_3 = \beta_2\left(\frac{w(1-y^2)}{1+w}\right)^{\beta_2-1}\left(\frac{-2wy}{1+w}\right)$.

Compare x^V to x^R, and y^V to y^R: We numerically solve for x^V and $\left(x^R, y^R\right)$ under various values of β_1 and β_2. We find that the general result that $x^R < x^V$ and $y^R \leq y^V$ continues to hold. Figure 8 illustrates a representative comparison result.



(a) x^V vs. x^R



(b) y^V vs. y^R

Figure 8: Equilibrium Comparison ($\beta_1 = 0.85$, $\beta_2 = 1.15$, $c = 0.15$, $w = 0.25$)

S2.3 Normal Distribution

In this section, we let ξ and ε follow normal distributions so as to extend the support from finite to infinite and the distribution from uniform to another common family. Specifically, $\xi \sim N\left(\kappa_1, \tau_1^2\right)$, and $\varepsilon \sim N\left(\kappa_2, \tau_2^2\right)$. Denote their cdf's and pdf's as $G(\xi)$ $(F(\varepsilon))$ and $g(\xi)$ $(f(\varepsilon))$, respectively.

Solve for x^V and y^V: Substituting in$\Pr\{V=1|\theta\} = 1 - G\left(-\mu_\theta\left(x+y\right)\right)$, we can write the expected payoff of the service provider as

$$E\pi = \alpha_0^V + \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left[2 - G\left(-x-y\right) - G\left(-\mu(x+y)\right)\right] - cx^2.$$

Taking the first derivative with respect to x and y, we have

$$\begin{aligned}
\frac{\partial E\pi}{\partial x} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left[g\left(-x-y\right) + \mu g\left(-\mu(x+y)\right)\right] - 2cx \\
\frac{\partial E\pi}{\partial y} &= \frac{1}{2}\left(\alpha_1^V - \alpha_0^V\right)\left[g\left(-x-y\right) + \mu g\left(-\mu(x+y)\right)\right]
\end{aligned}$$

Because $\frac{\partial E\pi}{\partial y} > 0$ for any y, therefore, $y^V = 1$. Meanwhile, x^V is the solution within $[0,1]$ to $K\left(x, y^V\right) = 2cx$, where $K\left(x,y\right) = \frac{1}{2}\left[\alpha_1^V\left(x,y\right) - \alpha_0^V\left(x,y\right)\right]\left[g\left(-x-y\right) + \mu g\left(-\mu(x+y)\right)\right]$.

Solve for x^R and y^R: Substituting in

$$\Pr\{R=1|\theta\} = 1 - F\left(wy^2\right) + \left[F\left(wy^2\right) - F\left(wy^2-1\right)\right]\left[1 - G\left(-\mu_\theta\left(x+y\right)\right)\right],$$

we can derive the expected payoff of the service provider as

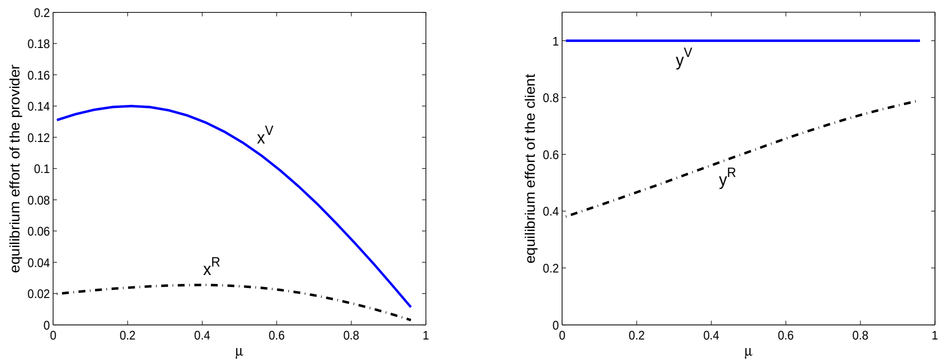
$$E\pi = \alpha_0^R + \frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\left\{2 - 2F\left(wy^2-1\right) - \left[F\left(wy^2\right) - F\left(wy^2-1\right)\right]\left[G\left(-x-y\right) + G\left(-\mu(x+y)\right)\right]\right\}.$$

Based on the first order condition, $\left(x^R, y^R\right)$ is the solution within $[0,1]$ to the two simultaneous equations: $\frac{\partial E\pi}{\partial x} = 0$ and $\frac{\partial E\pi}{\partial y} = 0$, which can be derived as

$$\begin{cases}
\frac{1}{2}\left(\alpha_1^R - \alpha_0^R\right)\left[F\left(wy^2\right) - F\left(wy^2-1\right)\right]D_1 = 2cx \\
2D_3 + \left(D_2 - D_3\right)\left[G\left(-x-y\right) + G\left(-\mu(x+y)\right)\right] - \left[F\left(wy^2\right) - F\left(wy^2-1\right)\right]D_1 = 0,
\end{cases}$$

where $D_1 = g(-x-y) + \mu g(-\mu(x+y))$, $D_2 = f\left(wy^2\right) 2wy$ and $D_3 = f\left(wy^2 - 1\right) 2wy$.

Compare x^V to x^R, and y^V to y^R: We numerically solve for x^V and $\left(x^R, y^R\right)$ under various normal distributions of ξ and ε. We find that the general result that $x^R < x^V$ and $y^R \leq y^V$ continues to hold. Figure 9 illustrates a representative comparison result.



(a) x^V vs. x^R (b) y^V vs. y^R

Figure 9: Equilibrium Comparison ($\kappa_1 = \kappa_2 = 0$, $\tau_1 = \tau_2 = 0.6$, $c = 0.15$, $w = 0.25$)

S3 Results When c is Small

In the paper, we focus on the generic case in which the service provider's cost coefficient c is not too small. As we discuss in the paper, doing so allows us to focus on the nontrivial regions with unique equilibria and to avoid the distraction of technical discussions. In this section, we complete the picture and characterize the equilibrium solutions when c is very small. As we show, our main results extend to the case of small c robustly.

We follow the baseline model and characterize the equilibrium results when c is very small. In the case when the service outcome serves as a signal, we can solve the equilibrium effort in a closed form and hence fully derive all possible equilibria under explicit conditions.

Proposition A.1. *When the market observes the outcome V alone as a signal, for $c \leq \frac{1}{8}$, the equilibrium effort of the client is always $y^V = 1$; as for the equilibrium effort of the provider, (i) when $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{8}$ and $\mu > \frac{1}{3}$, there are three equilibria: $x^V = 1$, $\frac{c(3-\mu) \pm \sqrt{c^2(3-\mu)^2 - c(1-\mu^2)}}{2c(1+\mu)}$; (ii)* *when* $c = \frac{1-\mu^2}{(3-\mu)^2}$ *and* $\mu > \frac{1}{3}$*, there are two equilibria:* $x^V = 1,\ \frac{3-\mu}{2(1+\mu)}$*; (iii) when* $c = \frac{1}{8}$ *and* $\mu > \frac{1}{3}$*, there are two equilibria:* $x^V = 1,\ \frac{2(1-\mu)}{1+\mu}$*; (iv) otherwise, there is only one equilibrium:* $x^V = 1$*.*

Proof. Following the proof of Proposition 2, we have the first order derivative with respect to y always be a positive constant. Therefore, $y^V = 1$. Recall that the first derivative with respect to x equals $K(x) - C(x)$, where $K(x) = \frac{1+\mu}{4}\triangle\alpha^V\left(x, y^V\right) = \frac{1-\mu}{2[3-\mu-(1+\mu)x]}$ and $C(x) = 2cx$. As we have shown, $K(0) > C(0) = 0$. When $c \leq \frac{1}{8}$, we have $K(1) \geq C(1)$, so $\frac{\partial E\pi}{\partial x}|_{x=1} = K(1) - C(1) \geq 0$. As a result, the corner solution at the upper bound, $x^V = 1$, is always an equilibrium solution. In addition, when $c > \frac{1-\mu^2}{(3-\mu)^2}$, $K(x)$ and $C(x)$ intersect twice, i.e., $K(x) = C(x)$ yields two solutions $\frac{c(3-\mu)\pm\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)}$. Note that $\frac{1-\mu^2}{(3-\mu)^2} \leq \frac{1}{8}$ for all $\mu \in [0,1]$ and the strict inequality holds if $\mu \neq \frac{1}{3}$. Also, given $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{8}$, $\frac{c(3-\mu)+\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)} < 1$ if and only if $\mu > \frac{1}{3}$, and $\frac{c(3-\mu)-\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)} > 1$ if and only if $\mu < \frac{1}{3}$. As a result, when $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{8}$ and $\mu > \frac{1}{3}$, there are three equilibrium x^V's within $[0,1]$, $x^V = 1$, $\frac{c(3-\mu)\pm\sqrt{c^2(3-\mu)^2-c(1-\mu^2)}}{2c(1+\mu)}$, which corresponds to (i). When $c = \frac{1-\mu^2}{(3-\mu)^2}$, $K(x)$ and $C(x)$ intersect once at $\frac{3-\mu}{2(1+\mu)}$. Notice that $\frac{3-\mu}{2(1+\mu)} < 1$ if and only if $\mu > \frac{1}{3}$. Therefore, when $c = \frac{1-\mu^2}{(3-\mu)^2}$ and $\mu > \frac{1}{3}$, there are two equilibrium x^V's within $[0,1]$, $x^V = 1$, $\frac{3-\mu}{2(1+\mu)}$, which corresponds to (ii). When $\frac{1-\mu^2}{(3-\mu)^2} < c = \frac{1}{8}$, $K(x) = C(x)$ yields two solutions: 1 and $\frac{2(1-\mu)}{1+\mu}$. Notice that $\frac{2(1-\mu)}{1+\mu} < 1$ if and only if $\mu > \frac{1}{3}$. Therefore, when $c = \frac{1}{8}$ and $\mu > \frac{1}{3}$, there are two equilibrium x^V's within $[0,1]$, $x^V = 1$, $\frac{2(1-\mu)}{1+\mu}$, which corresponds to (iii). In all the other cases (i.e., $c < \frac{1-\mu^2}{(3-\mu)^2}$, or $\mu \leq \frac{1}{3}$ and $c \leq \frac{1}{8}$), there is only one equilibrium, $x^V = 1$, which corresponds to (iv). □

When the review serves as a signal, the equilibrium effort does not have a closed form. As a result, fully explicit conditions for different equilibria are not available. Nevertheless, we can characterize the equilibria in general and provide sufficient conditions under which multiple equilibria exist.

Proposition A.2. *When the market observes the review* R *alone as a signal, for* $c \leq \frac{1}{2}M(1)$*, the equilibrium effort of the client is always* $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$*; as for the equilibrium effort of the provider, one, two, or three equilibria could exist, among which* $x^R = 1$ *is always an equilibrium, and the other possible equilibria must be the solution(s) to* $M(x) = 2cx$ *within the range of* $x \in [0,1]$*. In particular, when* $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{2}M(1)$ *and* $\mu > \frac{1}{3}$*, there are three equilibrium* x^R*'s. Here,* $M(x) \equiv \frac{1+\mu}{4(1+w)}\Delta\alpha^R\left(x, y^R\right)$*.*

Proof. Following the proof of Proposition 2, we have the first order derivative with respect to y as $\frac{\partial E\pi}{\partial y} = \frac{\triangle\alpha^R(x^R,\, y^R)}{4(1+w)}(1+\mu-8wy)$. It is thus easy to conclude that $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$. Recall that the first derivative with respect to x equals $M(x) - C(x)$, where $M(x) = \frac{1+\mu}{4(1+w)}\Delta\alpha^R(x, y^R)$ and $C(x) = 2cx$. As we have shown, $M(0) > C(0) = 0$. When $c \leq \frac{1}{2}M(1)$, we have $M(1) \geq C(1)$, so $\frac{\partial E\pi}{\partial x}|_{x=1} = M(1) - C(1) \geq 0$. As a result, the corner solution at the upper bound, $x^R = 1$, is always an equilibrium solution. In addition, because $M(x) - C(x) = 0$ simplifies to a cubic equation of x (with at most three real roots), and we have $M(0) > C(0)$ and $M(1) \geq C(1)$, we can conclude that $M(x)$ and $C(x)$ intersect at most twice within $[0,1]$. As a result, within the range of $x \in [0,1]$, $M(x) = C(x)$ may yield no solution (i.e., $M(x) > C(x)$ for $\forall x \in [0,1]$), one solution (i.e., $M(x)$ intersects $C(x)$ at a point where $M(x)$ is tangent to $C(x)$), or two solutions (i.e., $M(x)$ intersects $C(x)$ twice). Consequently, including the corner solution $x^R = 1$, there could exist one, two, or three equilibrium x^R's.

To derive a sufficient condition under which multiple equilibria exist, recall the results in Proposition A.1 that when $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{8}$ and $\mu > \frac{1}{3}$, $K(x)$ intersects $C(x)$ twice within $x \in (0,1)$. Notice that when $w = 0$, $M(x|w=0) = K(x)$. By the proof of Corollary 2, $M(x)$ is decreasing in w. Therefore, we have $M(x) < K(x)$ for any $w > 0$. We know that $M(0) > C(0)$, and $M(1) > C(1)$ when $c < \frac{1}{2}M(1) \left(\leq \frac{1}{8}\right)$. Therefore, we can conclude that when $\frac{1-\mu^2}{(3-\mu)^2} < c < \frac{1}{2}M(1)$ and $\mu > \frac{1}{3}$, $M(x)$ intersects $C(x)$ twice within $x \in (0,1)$, yielding two solutions of x^R. Consequently, including the corner solution $x^R = 1$, there are three equilibrium x^R's under this condition. □

Next, we compare the equilibrium effort under the two scenarios. Given the possible multiple equilibria, we need a certain criterion to refine the equilibrium concept and eliminate those less "reasonable" equilibria. We consider that among multiple possible equilibria, only the one that yields the highest expected payoff for the service provider sustains. Under such a criterion, we are able to narrow down to a single equilibrium for each scenario when c is small. The next proposition shows that the equilibrium comparison results in the paper extend robustly to the case when c is small.

Proposition A.3. *The equilibrium effort levels of both the service provider and the client are lower when the review is observed as a signal than when the outcome is observed as a signal. Specifically, for $c \leq \frac{1}{8}$, $x^R \leq x^V$, and $y^R \leq y^V$ with strict inequality when $w > \frac{1+\mu}{8}$.*

Proof. Recall that $y^V = 1$ and $y^R = \min\left\{\frac{1+\mu}{8w}, 1\right\}$. Therefore, $y^R \leq y^V$ with strict inequality when $w > \frac{1+\mu}{8}$.

Next, we compare x^R and x^V. Following the same analysis as in (A.32), we can show that the ex ante payoff for the service provider equals $\Pr(\theta = H) - c(x^*)^2$. As a result, the equilibrium effort x^* that yields the highest expected payoff for the provider is the one that is the lowest in value. In other words, under such a refinement criterion, for $c \leq \frac{1}{8}$, the equilibrium effort of the provider, x^R (x^V), is either the smaller solution to $M(x) = 2cx$ ($K(x) = 2cx$) within $(0, 1)$ (if two solutions exist), the only solution to $M(x) = 2cx$ ($K(x) = 2cx$) within $(0, 1)$ (if only one solution exists), or the upper bound 1 (if no such solution exists). Notice in particular that for $\frac{1}{2}M(1) < c \leq \frac{1}{8}$, we have shown in the paper that there is a unique solution x^R to $M(x) = 2cx$.

Recall that $M(x|w = 0) = K(x)$, and $M(x)$ is decreasing in w, whereas $K(x)$ is independent of w. We thus have $M(x) < K(x)$ for any $w > 0$. For $c \leq \frac{1}{8}$, if $K(x)$ intersects $2cx$ within $(0, 1)$, then x^V is the (smaller) solution to $K(x) = 2cx$, and therefore $K\left(x^V\right) = 2cx^V$. Because $M(0) > 2c \cdot 0$ and $M\left(x^V\right) < K\left(x^V\right) = 2cx^V$, $M(x)$ must intersect $2cx$ at $x < x^V$. Consequently, as the (smaller) solution to $M(x) = 2cx$, x^R must be less than x^V. On the other hand, if $K(x)$ does not intersect $2cx$ within $(0, 1)$, then x^V takes the upper bound 1. As a result, $x^R \leq x^V = 1$. Altogether, we can conclude that $x^R \leq x^V$ for $c \leq \frac{1}{8}$. ◻

S4 Results for Section 6.2

We follow the baseline model with the only difference that the client's effort y is now observed by the market when the review R serves as a signal. As we discuss in the paper, we can derive the perfect Bayesian equilibrium when R serves as a signal as follows.

Given any chosen y, the market forms its rational belief about the true type of the service provider based on the observation of the realized value of R and the client's effort level y, while rationally anticipating the equilibrium effort of the provider $\dot{x}^R$, such that

$$\begin{aligned} \alpha_1^R &= \frac{\Pr(R = 1|\theta = H)\Pr(\theta = H)}{\Pr(R = 1|\theta = H)\Pr(\theta = H) + \Pr(R = 1|\theta = L)\Pr(\theta = L)} \\ &= \frac{2w\left(1 - y^2\right) + \left(\dot{x}^R + y\right)}{4w\left(1 - y^2\right) + (1 + \mu)\left(\dot{x}^R + y\right)}, \end{aligned} \tag{A.26}$$

and

$$\begin{aligned}
\alpha_0^R &= \frac{\Pr(R=0|\theta=H)\Pr(\theta=H)}{\Pr(R=0|\theta=H)\Pr(\theta=H)+\Pr(R=0|\theta=L)\Pr(\theta=L)} \\
&= \frac{2(1+w)-2w\left(1-y^2\right)-\left(\dot{x}^R+y\right)}{4(1+w)-4w\left(1-y^2\right)-(1+\mu)\left(\dot{x}^R+y\right)}. \qquad \text{(A.27)}
\end{aligned}$$

On the other hand, rationally anticipating the market belief α_1^R and α_0^R, the service provider chooses $\dot{x}^R$ such that it maximizes her expected payoff

$$\begin{aligned}
E\pi &= \Pr(\theta=H)\left[\Pr(R=1|\theta=H)\cdot\alpha_1^R+\Pr(R=0|\theta=H)\cdot\alpha_0^R\right] \\
&\quad +\Pr(\theta=L)\left[\Pr(R=1|\theta=L)\cdot\alpha_1^R+\Pr(R=0|\theta=L)\cdot\alpha_0^R\right]-cx^2 \\
&= \frac{1}{2}\left[\alpha_0^R+\Pr(R=1|\theta=H)\left(\alpha_1^R-\alpha_0^R\right)\right]+\frac{1}{2}\left[\alpha_0^R+\Pr(R=1|\theta=L)\left(\alpha_1^R-\alpha_0^R\right)\right]-cx^2 \\
&= \alpha_0^R+\Delta\alpha^R\left[\frac{w}{1+w}\left(1-y^2\right)+\frac{1}{1+w}\frac{1+\mu}{4}(x+y)\right]-cx^2, \qquad \text{(A.28)}
\end{aligned}$$

where $\Delta\alpha^R=\alpha_1^R-\alpha_0^R$. As a result,

$$\dot{x}^R(y)=\arg\max_x\ \alpha_0^R\left(\dot{x}^R,y\right)+\Delta\alpha^R\left(\dot{x}^R,y\right)\left[\frac{w}{1+w}\left(1-y^2\right)+\frac{1}{1+w}\frac{1+\mu}{4}(x+y)\right]-cx^2. \quad \text{(A.29)}$$

The first derivative of (A.29) with respect to x yields (note that $\dot{x}^R$ inside α_0^R and α_1^R is treated as a constant here)

$$\frac{\partial E\pi}{\partial x}=\Delta\alpha^R\left(\dot{x}^R,y\right)\frac{1+\mu}{4(1+w)}-2cx=0 \qquad \text{(A.30)}$$

Therefore, the equilibrium effort of the provider (given any chosen y), $\dot{x}^R(y)$, is the solution within $[0,1]$ to the equation

$$M\left(\dot{x}^R,y\right)=2c\dot{x}^R, \qquad \text{(A.31)}$$

where $M\left(\dot{x}^R,y\right)=\Delta\alpha^R\left(\dot{x}^R,y\right)\frac{1+\mu}{4(1+w)}$. As we can see, (A.31) follows the same form as previously. As a result, the equilibrium effort levels $\dot{x}^R$ and $\dot{y}^R$ follow the same functional relationship as before, and all the relevant properties proved in the baseline model (e.g., the existence and uniqueness of the solution to (A.31)) continue to hold.

Anticipating the equilibrium $\dot{x}^R(y)$, the service provider chooses y such that it maximizes her expected payoff $E\pi$ as in (A.28). Notice that when we substitute back the equilibrium effort $\dot{x}^R(y)$, because the market belief is rational, the provider's expected payoff simplifies to a constant term minus her effort cost. Specifically, we have

$$
\begin{aligned}
E\pi &= \Pr(\theta=H)\left[\Pr(R=1|\theta=H)\cdot\alpha_1^R+\Pr(R=0|\theta=H)\cdot\alpha_0^R\right] \\
&\quad +\Pr(\theta=L)\left[\Pr(R=1|\theta=L)\cdot\alpha_1^R+\Pr(R=0|\theta=L)\cdot\alpha_0^R\right]-c\dot{x}^R(y)^2 \\
&= \left[\Pr(\theta=H)\Pr(R=1|\theta=H)+\Pr(\theta=L)\Pr(R=1|\theta=L)\right]\alpha_1^R \\
&\quad +\left[\Pr(\theta=H)\Pr(R=0|\theta=H)+\Pr(\theta=L)\Pr(R=0|\theta=L)\right]\alpha_0^R-c\dot{x}^R(y)^2 \\
&= \Pr(\theta=H)\Pr(R=1|\theta=H)+\Pr(\theta=H)\Pr(R=0|\theta=H)-c\dot{x}^R(y)^2 \\
&= \Pr(\theta=H)-c\cdot\dot{x}^R(y)^2. \qquad \text{(A.32)}
\end{aligned}
$$

The third equality holds by substituting in the definition of α_1^R and α_0^R. As a result, the optimal y is the one that minimizes $\dot{x}^R(y)$, that is,

$$\dot{y}^R=\arg\min_y \dot{x}^R(y). \qquad \text{(A.33)}$$

Consequently, the equilibrium effort level of the provider herself equals $\dot{x}^R\left(\dot{y}^R\right)$.

Having deriving both $\dot{x}^R$ and $\dot{y}^R$ under the extended model, we can compare them with the equilibrium effort levels when the outcome V serves as a signal, x^V and y^V, which remains the same. We have the similar result as follows.

Proposition A.4. *When the client's effort is observable to the market when the review serves as a signal, $\dot{x}^R<x^V$ and $\dot{y}^R\le y^V$.*

Proof. By (A.33), because $\dot{y}^R$ minimizes $\dot{x}^R(y)$ over all possible y's, we have $\dot{x}^R\left(\dot{y}^R\right)\le\dot{x}^R\left(y^R\right)$. Notice that $\dot{x}^R\left(y^R\right)$ is simply the equilibrium x^R derived under the baseline model, and it is shown that $x^R<x^V$. Therefore, $\dot{x}^R\left(\dot{y}^R\right)<x^V$. Recall that $y^V\equiv 1$, and $\dot{y}^R\le 1$. Therefore, we have $\dot{y}^R\le y^V$. □

Next, we further extend the case when the client incorporates private information into the review. We focus on the case of symmetric private signal with $\sigma_H=\gamma$ and $\sigma_L=1-\gamma$, $\gamma\in\left[\frac{1}{2},1\right]$. We now consider that the client's effort y is observable to the market in this case.

Following similar reasoning, we can derive that the equilibrium effort of the provider $\dot{\tilde{x}}^R(y)$ is the unique solution to $\tilde{M}\left(\dot{\tilde{x}}^R,y\right)=2c\dot{\tilde{x}}^R$, where $\tilde{M}(x,y)=\Delta\tilde{\alpha}^R(x,y)\frac{\gamma+(1-\gamma)\mu}{4(1+w)}$, and $\dot{\tilde{y}}^R$ minimizes

$\dot{\tilde{x}}^R(y)$. In order to compare $\dot{\tilde{x}}^R$ and $\dot{\tilde{y}}^R$ with x^V and y^V as γ varies, we first prove a useful lemma.

Lemma A.6. *$\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R\right)$ is increasing in γ.*

Proof. First notice that $\frac{d}{d\gamma}\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R(\gamma);\gamma\right) = \frac{\partial}{\partial\gamma}\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma\right)$ by Envelope Theorem. Because $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma\right)$ is defined by the implicit function $F\left(\dot{\tilde{x}}^R,\dot{\tilde{y}}^R;\gamma\right) \equiv \tilde{M}\left(\dot{\tilde{x}}^R,\dot{\tilde{y}}^R;\gamma\right) - 2c\dot{\tilde{x}}^R = 0$. Therefore,

$$\frac{\partial}{\partial\gamma}\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma\right) = -\frac{\frac{\partial F}{\partial\gamma}}{\frac{\partial F}{\partial x}} = -\frac{\frac{\partial}{\partial\gamma}\tilde{M}\left(\dot{\tilde{x}}^R,\dot{\tilde{y}}^R;\gamma\right)}{\frac{\partial}{\partial x}\tilde{M}\left(\dot{\tilde{x}}^R,\dot{\tilde{y}}^R;\gamma\right) - 2c}$$

We have shown in the proof of Lemma A.5 that $\frac{\partial}{\partial\gamma}\tilde{M}(x,y;\gamma) > 0$ for any x and y. Also, we know that $\frac{\partial}{\partial x}\tilde{M}\left(\dot{\tilde{x}}^R,\dot{\tilde{y}}^R;\gamma\right) < 2c$ (by the condition of the crossing of $\tilde{M}(x)$ and $2cx$ at $\dot{\tilde{x}}^R$). Therefore, $\frac{\partial}{\partial\gamma}\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma\right) > 0$, and hence $\frac{d}{d\gamma}\dot{\tilde{x}}^R > 0$, that is, $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R\right)$ is increasing in γ. $\square$

We then show a similar result regarding the relative magnitude of $\dot{\tilde{x}}^R$ and x^V as γ changes in the extended model.

Proposition A.5. *There exists (c,w,μ) such that for a certain threshold $\tilde{\gamma}(c,w,\mu)$, $\dot{\tilde{x}}^R < x^V$ if $\gamma < \tilde{\gamma}$; $\dot{\tilde{x}}^R > x^V$ if $\gamma > \tilde{\gamma}$.*

Proof. Following similar reasoning as in the proof of Proposition A.4, when $\gamma = \frac{1}{2}$, $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma=\frac{1}{2}\right) \leq \tilde{x}^R\left(\tilde{y}^R;\gamma=\frac{1}{2}\right)$. We have already shown in the paper that $\tilde{x}^R\left(\tilde{y}^R;\gamma=\frac{1}{2}\right) < x^V$. Therefore, $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma=\frac{1}{2}\right) < x^V$. Next, we show that when $\gamma = 1$, there exists (c,w,μ) such that $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma=1\right) > x^V$.

Consider $w = \frac{1}{4}$. Recall that

$$\tilde{M}(x,y;\gamma=1) = \frac{1}{2(4-x-y)+4w(1+y^2)}. \tag{A.34}$$

Notice that $\tilde{M}(x,y;\gamma=1)$ is increasing in y when $y < \frac{1}{4w}$. For $w = \frac{1}{4}$, $\tilde{M}(x,y;\gamma=1)$ is increasing in y for $\forall y \in [0,1]$. Consequently, $\dot{\tilde{x}}^R(y)$ is increasing in y for all y. As a result, $\dot{\tilde{y}}^R = 0$, which minimizes $\dot{\tilde{x}}^R(y)$. Therefore, $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R\right) = \frac{1}{9c+\sqrt{c(81c-4)}} > 0$. Recall that $x^V = \frac{c\left(4-(1+\mu)y^V\right)-\sqrt{c^2(4-(1+\mu)y^V)^2-c(1-\mu^2)}}{2c(1+\mu)}$ is decreasing in μ. When $\mu = 1$, $x^V = 0 < \dot{\tilde{x}}^R$. Therefore, there exists a cutoff $\tilde{\mu}$ such that when $\mu > \tilde{\mu}$, $x^V < \dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R\right)$.

Now that we have shown there exists (c,w,μ) such that $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma=1\right) > x^V$, and $\dot{\tilde{x}}^R\left(\dot{\tilde{y}}^R;\gamma=\frac{1}{2}\right) <$

x^V always holds, given $\dot{\hat{x}}^R$ $(\dot{\hat{y}}^R)$ is increasing in γ by Lemma A.6 and x^V is independent of γ, we arrive at the said conclusion. $\square$