Online Appendix

Forward-Looking Behavior in Mobile Data Consumption and Targeted Promotion Design: A Dynamic Structural Model

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A Proofs of Propositions in Section 5.1

To simplify notations, without causing confusion, below we suppress subscripts i and t. Also notice that throughout all the proofs below, wherever *strict* monotonicity or concavity applies, we explicitly stress it; without explicit stress of strictness, we mean weak monotonicity/concavity.

Proof of Proposition 1. Myopic users determine their daily usage by maximizing the per-period utility only, which is defined in (1). The optimal daily usage a^* can thus be derived as

$$a^* = \begin{cases} \mu + \xi - \eta p \, (> q) & \text{if } 0 \leq q < \mu + \xi - \eta p \\ q & \text{if } \mu + \xi - \eta p \leq q \leq \mu + \xi \\ \max\{\mu + \xi, 0\} \, (< q) & \text{if } q > \mu + \xi \end{cases} \tag{A.1}$$

In any day before the day when the data plan quota is fully expended, $a^* = \max\{\mu + \xi, 0\} \, (< q)$, which is obviously independent of q. Q.E.D.

In order to prove Proposition 2, we first prove two key lemmas with regard to the properties of the expected value function $\bar{V}(q, d)$ as defined in (5).

Lemma A.1. *For the last period, the expected value function $\bar{V}(q, d = 1)$ is continuous, increasing, differentiable, and strictly concave in the remaining data plan quota q.*

Proof. Recall that in the last period,

$$V(q, d = 1, \xi) = \max_{a \geq 0} \left[(\mu + \xi) a - \frac{1}{2} a^2 - \eta p \max\{a - q, 0\} \right] \tag{A.2}$$

We can thus explicitly solve the value function as

$$V(q, d=1, \xi) = \begin{cases} \frac{1}{2}(\mu+\xi-\eta p)^2 + \eta p q & \text{if } 0 \le q < \mu+\xi-\eta p \\ (\mu+\xi)q - \frac{1}{2}q^2 & \text{if } \mu+\xi-\eta p \le q \le \mu+\xi \\ \frac{1}{2}(\max\{\mu+\xi, 0\})^2 & \text{if } q > \mu+\xi, \end{cases} \tag{A.3}$$

where $q \ge 0$. It is easy to show that $V(q, d=1, \xi)$ is continuous, increasing, differentiable, and concave in q given any ξ. The continuity can be easily verified by checking the function value at each endpoint. (Notice that because $q \ge 0$, if $\mu+\xi-\eta p < \mu+\xi < 0$, only the third segment applies and (A.3) reduces to a constant so that $V(q, d=1, \xi) \equiv 0$ for $\forall q \ge 0$; if $\mu+\xi-\eta p < 0 < \mu+\xi$, (A.3) reduces to two segments.) The monotonicity is immediate because the piecewise function is continuous and piecewise increasing in q. $V(q, d=1, \xi)$ is differentiable because the left and right derivatives are equal at each endpoint: $\frac{\partial}{\partial q}V_{q\to(\mu+\xi-\eta p)^+}(q, d=1, \xi) = \frac{\partial}{\partial q}V_{q\to(\mu+\xi-\eta p)^-}(q, d=1, \xi) = \eta p$, and $\frac{\partial}{\partial q}V_{q\to(\mu+\xi)^+}(q, d=1, \xi) = \frac{\partial}{\partial q}V_{q\to(\mu+\xi)^-}(q, d=1, \xi) = 0$. $V(q, d=1, \xi)$ is concave in q because it is differentiable and piecewise concave in q.

Given that $V(q, d=1, \xi)$ is continuous, increasing, and differentiable in q for any ξ, it is immediate that the expected value function $\bar{V}(q, d) = E_\xi V(q, d, \xi)$, as an integral over all ξ, is also continuous, increasing, and differentiable in q.

To show that $\bar{V}(q, d)$ is strictly concave in q, note that because $V(q, d=1, \xi)$ is concave in q, by the definition of concavity, for any $q_1, q_2 > 0$ and $\lambda \in (0, 1)$, we have

$$V((1-\lambda)q_1 + \lambda q_2, d=1, \xi) \ge (1-\lambda)V(q_1, d=1, \xi) + \lambda V(q_2, d=1, \xi) \tag{A.4}$$

for any ξ. Because $V(q, d=1, \xi)$ is *strictly* concave in the second segment in (A.3), *strict* inequality holds in (A.4) when $\xi \in (-\mu+q_1, -\mu+q_1+\eta p) \cup (-\mu+q_2, -\mu+q_2+\eta p)$. Recall that ξ is a random variable with a continuous support over the entire real field. Therefore, when taking expectation over ξ on both sides of (A.4), we have

$$E_\xi V((1-\lambda)q_1 + \lambda q_2, d=1, \xi) > (1-\lambda)E_\xi V(q_1, d=1, \xi) + \lambda E_\xi V(q_2, d=1, \xi), \tag{A.5}$$

which shows $\bar{V}(q, d=1)$ is *strictly* concave in q. Q.E.D. □

Lemma A.2. *If the expected value function for the next period, $\bar{V}\left(q^{'}, d-1\right)$, is continuous, increasing, differentiable, and strictly concave in $q^{'}$, then the expected value function for the current period, $\bar{V}(q,d)$, is also continuous, increasing, differentiable, and strictly concave in q.*

Proof. We first show that given any ξ, the value function $V(q,d,\xi)$ is continuous, increasing, differentiable, and concave in q if $\bar{V}\left(q^{'}, d-1\right)$ is continuous, increasing, differentiable, and strictly concave in $q^{'}$. Substituting (1) and (3) into (4), we can rewrite the current-period value function as

$$V(q,d,\xi) = \max_{a\geq 0}\left[(\mu+\xi)a - \frac{1}{2}a^2 - \eta p[a-q]^+ + \beta\bar{V}\left([q-a]^+, d-1\right)\right], \tag{A.6}$$

where $[\cdot]^+$ stands for $\max\{\cdot,0\}$. To simplify notation, we use $\bar{V}_q(q,d)$ to represent $\frac{\partial}{\partial q}\bar{V}(q,d)$ for the rest of this proof.

Let $\tilde{a}$ be the solution to the first order condition (with respect to a) when $a<q$, that is,

$$\mu+\xi-\tilde{a}-\beta\bar{V}_q(q-\tilde{a}, d-1) = 0 \tag{A.7}$$

Therefore, $\tilde{a}<q$ if and only if $\mu+\xi-q-\beta\bar{V}_q(0,d-1)<0$. When $a>q$, the first order condition yields

$$\mu+\xi-a^*-\eta p = 0. \tag{A.8}$$

$a^*>q$ if and only if $\mu+\xi-q-\eta p>0$. Notice that $\beta\bar{V}_q(0,d-1)<\eta p$ given $\beta<1$. Therefore, we can summarize the optimal usage in the current period as

$$a^*(q,d,\xi) = \begin{cases} \mu+\xi-\eta p\,(>q) & \text{if } 0\leq q<\mu+\xi-\eta p \\ q & \text{if } \mu+\xi-\eta p\leq q\leq \mu+\xi-\beta\bar{V}_q(0,d-1) \\ \max\{\tilde{a},0\}\,(<q) & \text{if } q>\mu+\xi-\beta\bar{V}_q(0,d-1) \end{cases} \tag{A.9}$$

Again, because $q\geq 0$, if $\mu+\xi-\beta\bar{V}_q(0,d-1)<0$ or $\mu+\xi-\eta p<0$, (A.9) reduces to one or two segments only. Accordingly, the current-period value function can be written as

$$V(q,d,\xi)=\begin{cases}\frac{1}{2}(\mu+\xi-\eta p)^2+\eta pq+\beta\bar{V}(0,d-1) & \text{if } 0\le q<\mu+\xi-\eta p\\ (\mu+\xi)q-\frac{1}{2}q^2+\beta\bar{V}(0,d-1) & \text{if } \mu+\xi-\eta p\le q\le\mu+\xi-\beta\bar{V}_q(0,d-1)\\ F(q;\xi) & \text{if } q>\mu+\xi-\beta\bar{V}_q(0,d-1),\end{cases} \tag{A.10}$$

where $F(q;\xi)$ is defined by substituting the optimal usage $a^*=\max\{\tilde{a},0\}\,(<q)$ into (A.6), that is,

$$F(q;\xi)=(\mu+\xi)a^*-\frac{1}{2}a^{*2}+\beta\bar{V}(q-a^*,d-1). \tag{A.11}$$

It is easy to show that $V(q,d,\xi)$ is continuous in q by verifying the continuity of function value at the endpoints: for example, when $q=\mu+\xi-\beta\bar{V}_q(0,d-1)$, $a^*=q$ so $F(q;\xi)=(\mu+\xi)q-\frac{1}{2}q^2+\beta\bar{V}(0,d-1)$. To show that $V(q,d,\xi)$ is increasing in q, we just need to show $F(q;\xi)$ is increasing in q, because it is obviously true for the first two segments of (A.10). Taking derivative with respect to q on both sides of (A.11), by Envelope Theorem, we have

$$F_q(q;\xi)=\beta\bar{V}_q(q-a^*,d-1)\ge 0, \tag{A.12}$$

because $\bar{V}\left(q',d-1\right)$ is increasing in q'. Therefore, $F(q;\xi)$ is increasing in q; so is $V(q,d,\xi)$.

It is easy to show that $V(q,d,\xi)$ is differentiable in q, noticing that

$$\frac{\partial}{\partial q}V_{q\to\left(\mu+\xi-\beta\bar{V}_q(0,d-1)\right)^-}(q,d,\xi)=\beta\bar{V}_q(0,d-1) \tag{A.13}$$

$$\frac{\partial}{\partial q}V_{q\to\left(\mu+\xi-\beta\bar{V}_q(0,d-1)\right)^+}(q,d,\xi)=F_q(q;\xi)=\beta\bar{V}_q(0,d-1), \tag{A.14}$$

where (A.14) holds by (A.12) and the fact that $a^*=q$ when $q=\mu+\xi-\beta\bar{V}_q(0,d-1)$.

We next show that $V(q,d,\xi)$ is concave in q. It is obvious that $V(q,d,\xi)$ is concave when $0\le q<\mu+\xi-\eta p$ and strictly concave when $\mu+\xi-\eta p\le q\le\mu+\xi-\beta\bar{V}_q(0,d-1)$. Given that $V(q,d,\xi)$ is differentiable in q, therefore, we only need to show that $F(q;\xi)$ is (strictly) concave in q for $q>\mu+\xi-\beta\bar{V}_q(0,d-1)$.

We prove by the definition of concavity. Consider any $q_1,q_2\ge\max\left\{\mu+\xi-\beta\bar{V}_q(0,d-1),0\right\}$, let $\hat{q}_1 = q_1 - a^*(q_1)$ and $\hat{q}_2 = q_2 - a^*(q_2)$. In other words, we use $\hat{}$ to represent the remaining quota at the beginning of the next period as a result of the optimal amount of usage in the current period. Note that $0 \le \hat{q}_1 \le q_1$ and $0 \le \hat{q}_2 \le q_2$. Denote $\bar{q} = \lambda q_1 + (1-\lambda) q_2$ and $\bar{\hat{q}} = \lambda \hat{q}_1 + (1-\lambda)\hat{q}_2$ for $\forall \lambda \in (0,1)$. Clearly, $0 \le \bar{\hat{q}} \le \bar{q}$. In addition, define $U\left(q, q^{'}\right) = (\mu + \xi)\left(q - q^{'}\right) - \frac{1}{2}\left(q - q^{'}\right)^2$. It is easy to show that $U\left(q, q^{'}\right)$ is concave in $\left(q, q^{'}\right)$ because it is a quadratic function with a negative semidefinite Hessian matrix. Hence, $F(\bar{q}; \xi)$ from (A.11) can be rewritten as

$$\begin{aligned} F(\bar{q};\xi) &= U\left(\bar{q}, \hat{\bar{q}}\right) + \beta \bar{V}\left(\hat{\bar{q}}, d-1\right) \\ &\ge U\left(\bar{q}, \bar{\hat{q}}\right) + \beta \bar{V}\left(\bar{\hat{q}}, d-1\right) \\ &\ge \lambda U(q_1, \hat{q}_1) + (1-\lambda) U(q_2, \hat{q}_2) + \beta \bar{V}\left(\bar{\hat{q}}, d-1\right) \\ &> \lambda U(q_1, \hat{q}_1) + (1-\lambda) U(q_2, \hat{q}_2) + \beta\lambda \bar{V}(\hat{q}_1, d-1) + \beta(1-\lambda)\bar{V}(\hat{q}_2, d-1) \\ &= \lambda F(q_1;\xi) + (1-\lambda) F(q_2;\xi) \end{aligned} \tag{A.15}$$

The first inequality in (A.15) holds because of the optimality of $\hat{\bar{q}}$; the second inequality holds because of the concavity of $U\left(q, q^{'}\right)$ in $\left(q, q^{'}\right)$; the third (strict) inequality holds because of the strict concavity of $\bar{V}\left(q^{'}, d-1\right)$ in $q^{'}$. As a result, $F(q;\xi)$ is strictly concave in q for any $q \ge \max\left\{\mu + \xi - \beta \bar{V}_q(0, d-1), 0\right\}$. Therefore, $V(q, d, \xi)$ is concave in q for any $q \ge 0$ and strictly concave if $q \ge \mu + \xi - \eta p$.

Given we have shown that $V(q, d, \xi)$ is continuous, increasing, differentiable, and concave in q for any ξ, and it is strictly concave in q when $\xi < -\mu + q + \eta p$, following the same logic as the last part of the proof of Lemma A.1, we conclude that $\bar{V}(q, d) = E_\xi V(q, d, \xi)$ is continuous, increasing, differentiable, and strictly concave in q. Q.E.D. □

Proof of Proposition 2. Recall the optimal usage $a^*(q, d, \xi)$ derived in (A.9) for any $d \ge 2$. In any day before the day when the data plan quota is fully expended, $a^*(q, d, \xi) = \max\{\tilde{a}, 0\} (< q)$. We want to show that $\tilde{a}$, the solution to (A.7), is *strictly* increasing in q.

Recall that $\tilde{a}(q)$ solves the first order condition

$$\mu + \xi - \tilde{a}(q) - \beta \bar{V}_q(q - \tilde{a}(q), d-1) = 0 \tag{A.16}$$

By Lemma A.1 and Lemma A.2, the expected value function $\bar{V}(\cdot,\cdot)$ is increasing and strictly concave in the remaining quota for any period. Therefore, for any $q^{'} > q$,

$$\mu + \xi - \tilde{a}(q) - \beta \bar{V}_q \left(q' - \tilde{a}(q), d-1\right) > 0 \tag{A.17}$$

because $\bar{V}_q (q' - \tilde{a}(q), d-1) < \bar{V}_q (q - \tilde{a}(q), d-1)$ given the strict concavity of $\bar{V}(\cdot, d-1)$. As a result, $\tilde{a}\left(q^{'}\right) > \tilde{a}(q)$. Therefore, $\tilde{a}$ is strictly increasing in q, which implies $a^{*}(q, d, \xi) = \max\{\tilde{a}, 0\}$ is strictly increasing in q if $0 < a^{*} < q$. Q.E.D.